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ON CYCLOTOMIC NUMBERS OF ORDER SIXTEEN

EMMA LEHMER

It has been shown by Dickson (1) that if $(i, j)_8$ is the number of solutions of

$$g^{8\nu+i} + 1 \equiv g^{8\mu+j} \pmod{p},$$

then $64(i, j)_8$ is expressible for each i, j, as a linear combination with integer coefficients of p, x, y, a, and b where

$$p = x^2 + 4y^2 = a^2 + 2b^2 = 8n + 1,$$

and

$$a \equiv b \equiv 1 \pmod{4},$$

while the sign of y and b depends on the choice of the primitive root g. There are actually four sets of such formulas depending on whether p is of the form $16n + 1$ or $16n + 9$ and whether 2 is a quartic residue or not.

We have recently (2) written out these formulas in detail and have shown that if 2 is not a quartic residue of $p = 16n + 1$ and if we define the ith class as the class containing t^i where

$$t \equiv \sqrt{2} \pmod{p},$$

then the sign of y is such that

$$\tfrac{1}{2}y \equiv -1 \pmod{4}$$

in the formulas for the cyclotomic constants $(i, j)_8$. The sign of b still remains in doubt.

The question has been raised by various people interested in the problem whether or not constants α, β, γ, δ, ϵ can be found such that

$$256\,(i, j)_{16} = p + \alpha x + \beta y + \gamma a + \delta b + \epsilon$$

at least for some i, j. To settle this problem the following experiment was undertaken on the SWAC.[1] Eight primes p of the form $32n + 1$ for which 2 is not a quartic residue were selected and the ith class was defined as before as the class containing t^i. (Since -1 is a 16-ic residue of such primes, there was no ambiguity of sign in choosing the square root.) The SWAC calculated the 51 independent cyclotomic constants for these eight primes. The remaining 205 constants can be obtained from these by the relations

$$(i, j)_{16} = (j, i)_{16} = (16 - i, j - i)_{16}.$$

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[1]The National Bureau of Standards Western Automatic Computer for the use of which we are grateful.

Then the constants of order eight could be obtained on the one hand from the relation

$$(i, j)_8 = (i, j)_{16} + (i + 8, j)_{16} + (i, j + 8)_{16} + (i + 8, j + 8)_{16},$$

and on the other from the formulas for $(i, j)_8$ in terms of x, y, a, b. Comparing these results, proper signs were affixed to b as well as x, y, and a. These decompositions will be found in the table, together with the 51 cyclotomic constants.

Finally, five of the eight primes, namely $p = 193, 449, 641, 769$, and 1409, were selected and an attempt was made to find a simultaneous solution of the five equations of the form

$$\alpha x_\nu + \beta y_\nu + \gamma a_\nu + \delta b_\nu + \epsilon = 256\, A_\nu - p_\nu$$

for $\nu = 1, 2, 3, 4$, and 5, where we let $(i, j)_{16} = A_\nu$ for these primes p_ν and for fixed i, j. We obtain

$$\begin{aligned}
-7\alpha + 6\beta - 11\gamma - 6\delta + \epsilon &= 256A_1 - 193,\\
-7\alpha - 10\beta + 21\gamma + 2\delta + \epsilon &= 256A_2 - 449,\\
25\alpha - 2\beta + 21\gamma - 10\delta + \epsilon &= 256A_3 - 641,\\
25\alpha + 6\beta - 11\gamma + 18\delta + \epsilon &= 256A_4 - 769,\\
25\alpha + 14\beta + 21\gamma + 22\delta + \epsilon &= 256A_5 - 1409.
\end{aligned}$$

Subtracting the first two and the last two we get

$$16\beta - 32\gamma - 8\delta = 256(A_1 - A_2) + 256$$

and

$$8\beta + 32\gamma + 4\delta = 256(A_5 - A_4) - 640.$$

Adding,

$$24\beta - 4\delta = 256(A_1 - A_2 - A_4 + A_5) - 384.$$

From the third and last of the original equations

$$16\beta + 32\delta = 256(A_5 - A_3) - 768.$$

Combining the last two we finally get

$$104\delta = 256(-2A_1 + 2A_2 - 3A_3 + 2A_4 + A_5) - 1536$$

or

$$13\delta = 32(-2A_1 + 2A_2 - 3A_3 + 2A_4 + A_5) - 192.$$

In order that δ be an integer it is necessary that

$$-2A_1 + 2A_2 - 3A_3 + 2A_4 + A_5 \equiv 6 \pmod{13}.$$

This condition is satisfied only if the constants A_ν stand for $(4, 8)_{16}$ and $(5, 10)_{16}$. In these two cases we get the tentative solution

$$256\,(4,8)_{16} = p - 271 - 10x + 8a + 16y$$

and

$$256\,(5,10)_{16} = p - 87 - 18x + 24a + 48y.$$

Unfortunately, neither of these proposed solutions holds for $p = 97$. Hence we must conclude that none of the cyclotomic constants $(i,j)_{16}$ is such that $256(i,j)_{16}$ is expressible as a linear combination with integer coefficients of p, a, b, x, y, in case $p = 32n + 1$, 2 not a quartic residue and the sign of b taken as consistent with the results on cyclotomic constants of order eight. Other hypotheses may be tested with the information provided by the SWAC, which is contained in the table.

The SWAC has also computed the cyclotomic constants of order sixteen for all other primes less than 1,000, as well as cyclotomic constants of order 24 for the same range.

A similar calculation was undertaken subsequently for primes of the form $32n + 17$. We chose $p = 241, 401, 433, 1009, 1297$, and, as before fixed the signs of b from the formulas giving the cyclotomic numbers of order eight. All these primes are such that t, given by

$$t^2 \equiv 2 \pmod p, \quad t < \tfrac{1}{2}(p-1),$$

is a non-residue.

The resulting equations are as follows:

$$\begin{aligned}
-15\alpha - 2\beta + 13\gamma + 6\delta + \epsilon &= 256A_1 - 241,\\
\alpha - 10\beta - 3\gamma - 14\delta + \epsilon &= 256A_2 - 401,\\
17\alpha + 6\beta - 19\gamma - 6\delta + \epsilon &= 256A_3 - 433,\\
-15\alpha + 14\beta - 19\gamma - 18\delta + \epsilon &= 256A_4 - 1009,\\
\alpha - 18\beta - 32\gamma + 6\delta + \epsilon &= 256A_5 - 1297.
\end{aligned}$$

Solving these for δ we get

$$9\delta = 16(2A_1 - 3A_2 + A_3 - A_4 + A_5).$$

Hence the expression $2A_1 - 3A_2 + A_3 - A_4 + A_5 \equiv 0 \pmod 9$ in order to have an integer solution. This was the case only when the A's stood for the cyclotomic constants $(0, 1)$, $(3, 2)$, and $(4, 0)$. In the first case β was not an integer, but the remaining two gave tentative solutions

$$256(3,2) = p - 15 + 6x - 8a + 16y$$

and

$$256(4,0) = p + 49 - 10x + 8a + 16y.$$

The first of these was easily disproved from the table for $p = 977$, while the second one held for $p = 977$ and it was necessary to calculate $(4, 0)$ for $p = 1361$. The calculations exhibited eight solutions, while the formula gave only six. So we must regretfully conclude that the cyclotomic constants of order 16 are not expressible in terms of these quadratic partitions alone.

The number (i, j) of solutions of $t^i x^{16} + 1 \equiv t^j y^{16}$ (mod p), $t^8 \equiv 2$ (mod p)

p	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, 7)	(0, 8)	(0, 9)	(0, 10)	(0, 11)	(0, 12)	(0, 13)	(0, 14)	(0, 15)	(1, 2)	(1, 3)
97	2	0	1	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
193	2	0	1	0	2	0	0	0	0	2	2	0	2	0	0	0	0	2
449	0	2	1	0	6	4	0	0	2	4	0	4	0	2	0	2	1	2
641	6	4	1	2	4	4	2	2	2	0	2	2	0	2	0	6	3	1
673	2	2	3	2	0	2	2	2	2	2	4	2	6	0	8	2	3	1
769	2	4	5	0	2	6	4	2	4	0	2	4	4	4	2	2	4	2
929	0	0	7	2	0	6	6	6	4	4	6	2	8	2	4	0	5	6
1409	0	8	5	10	4	10	6	6	8	4	0	4	4	6	10	2	7	3

p	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 8)	(1, 9)	(1, 10)	(1, 11)	(1, 12)	(1, 13)	(1, 14)	(2, 4)	(2, 5)	(2, 6)	(2, 7)	(2, 8)	(2, 9)	(2, 10)
97	1	1	0	1	1	1	0	0	0	0	1	0	0	1	0	0	0	0
193	1	1	1	1	0	1	1	2	0	1	1	1	0	0	1	0	1	3
449	2	2	1	4	1	3	1	1	1	3	1	3	2	1	0	3	3	1
641	2	0	0	2	4	2	1	6	1	3	2	4	5	4	2	2	3	2
673	2	3	2	4	3	3	3	3	3	1	4	1	3	4	0	1	3	1
769	3	2	3	3	4	4	0	0	6	2	5	2	4	3	3	2	2	4
929	4	5	5	4	4	3	2	3	3	6	3	1	2	3	2	4	4	3
1409	7	2	5	5	6	8	6	3	9	5	5	7	5	4	3	5	3	3

p	(2, 11)	(2, 12)	(2, 13)	(3, 6)	(3, 7)	(3, 8)	(3, 9)	(3, 10)	(3, 11)	(3, 12)	(4, 8)	(4, 9)	(4, 10)	(4, 11)	(5, 10)	x	y	a	b
97	2	1	0	1	0	0	0	0	1	1	0	0	1	0	0	9	−2	5	6
193	0	0	1	1	2	1	0	0	1	0	0	2	0	0	1	−7	6	−11	−6
449	3	2	2	3	0	2	2	1	1	2	1	0	3	1	2	−7	−10	21	2
641	2	4	1	1	4	4	3	3	3	3	1	2	3	4	2	25	−2	21	−10
673	5	1	3	3	5	3	1	3	4	4	4	1	3	0	4	−23	6	5	18
769	4	2	2	2	2	4	4	6	1	5	1	4	6	3	1	25	6	−11	18
929	5	1	7	2	4	4	2	5	2	5	5	2	4	2	4	−23	−10	−27	10
1409	8	8	5	3	5	5	7	9	4	6	5	4	8	5	8	25	14	21	22

The number (i, j) of solutions of $t^i x^{16} + 1 \equiv t^j y^{16}$ (mod p), $t^8 \equiv 2$ (mod p)

p	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, 7)	(0, 8)	(0, 9)	(0, 10)	(0, 11)	(0, 12)	(0, 13)	(0, 14)	(0, 15)	(1, 0)	(1, 1)
241	0	0	1	2	0	4	2	0	2	2	0	0	2	0	0	0	1	2
401	2	0	5	0	0	2	2	2	0	0	2	2	4	0	2	2	3	1
433	2	0	5	2	2	2	2	0	2	4	0	2	0	0	2	2	1	2
977	4	2	3	6	4	6	6	4	0	2	8	4	6	2	2	2	5	4
1009	4	2	9	4	4	2	4	0	2	8	2	4	4	2	6	6	2	4
1297	6	2	5	8	14	4	2	8	2	6	6	6	0	4	2	6	5	2

p	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 11)	(1, 12)	(1, 13)	(1, 14)	(1, 15)	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 13)
241	1	0	0	1	2	1	0	1	0	1	2	1	2	0	1	1	2	0	1
401	1	2	1	4	0	1	3	1	1	1	3	1	1	0	1	0	1	3	1
433	0	2	3	4	2	0	2	3	2	1	1	1	3	3	1	2	2	1	1
977	4	2	2	4	4	5	0	4	8	3	5	3	4	4	4	1	5	3	4
1009	3	6	6	4	4	2	6	7	4	2	3	2	8	6	1	4	5	3	5
1297	5	4	4	7	6	4	5	4	6	7	4	9	5	8	6	3	6	5	3

p	(2, 14)	(2, 15)	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 15)	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(5, 2)	x	y	a	b
241	2	0	1	1	0	0	1	0	1	2	0	1	1	3	−15	−2	13	6
401	2	4	2	1	1	2	1	2	0	1	2	1	3	2	1	−10	−3	−14
433	1	5	1	1	3	2	0	0	2	1	2	2	1	1	17	6	−19	−6
977	4	3	2	4	9	3	2	3	4	5	4	4	1	6	−31	−2	−3	22
1009	2	6	3	3	5	5	1	3	3	5	4	3	4	3	−15	14	−19	−18
1297	2	6	2	6	5	5	8	4	6	3	4	6	6	5	1	−18	−35	6

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1. L. E. Dickson, *Cyclotomy, higher congruences and Waring's problem*, Amer. J. Math. 57 (1935), 391–242.
2. Emma Lehmer, *On the number of solutions of* $u^k + D \equiv w^2 \pmod{p}$, to appear shortly in the Pacific Journal of Mathematics.

Berkeley, California

THE MAXIMAL PRIME DIVISORS OF LINEAR RECURRENCES

MORGAN WARD

1. Introduction. Let

$$(W): \quad W_0, W_1, \ldots, W_n, \ldots$$

be a linear integral recurring sequence of order $r > 2$; that is, a particular solution of the recurrence

$$\Omega_{n+r} = P_1 \Omega_{n+r-1} + P_2 \Omega_{n+r-2} + \ldots + P_r \Omega_n, \tag{1.1}$$

where $P_1, P_2, \ldots, P_r \neq 0$ are integers, and the initial values $W_0, W_1, \ldots, W_{r-1}$ are integers.

A positive integer m is said to be a *divisor* of (W) if it divides some term W_k with positive index k.

A prime number p is said to be *regular* in (W) if every power of p is a divisor of (W). If only a finite number of powers of p are divisors of (W), p is said to be *irregular*.

If there exist in (W) s consecutive terms divisible by p, say $W_k, W_{k+1}, \ldots, W_{k+s-1}$, but p never divides $s + 1$ consecutive terms of (W), p is said to be a divisor of (W) of order s, and k is said to be a zero of p in (W) of order s. Evidently s must be less than the order r of the recurrence. A prime of order s may have zeros in (W) of order less than s, and may be regular or irregular.

A prime divisor of (W) of the maximum possible order $r - 1$ will be called *maximal*.

I give in this paper a necessary condition that p shall be a maximal prime divisor of (W) under the assumption that the characteristic polynomial

$$f(z) = z^r - P_1 z^{r-1} - \ldots - P_r \tag{1.2}$$

of the recurrence has no repeated roots. When $r = 2$, all prime divisors of (W) which are not null divisors **(1)** are maximal, and the condition reduces to a criterion for a divisor due essentially to Marshall Hall **(2)** which is both necessary and sufficient. But if r is greater than two, the condition is no longer sufficient for p to be maximal in (W). In order for the condition to be sufficient the following additional restrictions on the recurrence and the prime must be made:

(i) $f(z)$ is of odd degree and irreducible;

(ii) The prime p is chosen so that $p - 1$ is prime to the degree r of $f(z)$;

(iii) $f(z)$ is irreducible modulo p.

As is shown in the concluding section of this paper, if these conditions fail to hold, the necessary condition for p to be maximal need no longer be sufficient.

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It will be evident from the sufficiency proof given under the restrictions just stated that if p is unramified in the root field of $f(z)$, a set of necessary and sufficient conditions can be stated in terms of the exponents to which a certain set of integers belong in the root field modulo all prime ideal factors of p. But these conditions appear too complicated to be of interest, and will not be developed here.

The results of the paper are stated as theorems in §4; the next two sections are devoted to algebraic and arithmetical preliminaries. The proofs are given in §§5, 6 and 7, and the concluding section is devoted to numerical examples.

2. Algebraic preliminaries. Let the characteristic polynomial $f(z)$ of the recurrence have r distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_r$ so that its discriminant D is not zero.

Then the general term of (W) is of the form

$$W_n = \beta_1 \alpha_1^n + \ldots + \beta_r \alpha_r^n \tag{2.1}$$

where the β are elements of the root-field $\mathfrak{R}$ of $f(z)$ to be specified presently.

Let $\Delta(W)$ denote the persymmetric determinant of order r in which the element in the ith row and jth column is W_{i+j-2}. The non-vanishing of $\Delta(W)$ is a necessary and sufficient condition that the recurring sequence (W) be of order r. Thus it easily follows from (2.1) that

$$\beta_1 \ldots \beta_r D = \Delta(W) \neq 0. \tag{2.2}$$

Define r polynomials $f_k(z)$ by $f_0(z) = 1$, $f_k(z) = z^k - P_1 z^{k-1} - \ldots - P_k$ $(k = 1, \ldots, r-1)$. Then the polynomial

$$w(z) = W_0 f_{r-1}(z) + W_1 f_{r-2}(z) + \ldots + W_{r-1} f_0(z)$$

has rational integral coefficients and is of degree less than r. Let

$$\gamma_i = w(\alpha_i) \qquad (i = 1, 2, \ldots, r).$$

Then the γ are integers in the root field $\mathfrak{R}$. Furthermore the polynomial

$$g(z) = (z - \gamma_1) \ldots (z - \gamma_r) = z^r - Q_1 z^{r-1} - \ldots - Q_r \tag{2.3}$$

has rational integral coefficients Q, and as we shall show in a moment, $Q_r \neq 0$.

Let $f'(z) = rz^{r-1} - (r-1) P_1 z^{r-2} - \ldots$ be the derivative of $f(z)$. Since $D = \pm f'(\alpha_1) \ldots f'(\alpha_r)$, none of the numbers $f'(\alpha)$ is zero. Furthermore it turns out that

$$\beta_i = \frac{\gamma_i}{f'(\alpha_i)} \qquad (i = 1, 2, \ldots, r).$$

Hence by (2.2), no γ is zero so that $Q_r \neq 0$, and

$$W_n = \frac{\gamma_1 \alpha_1^n}{f'(\alpha_1)} + \ldots + \frac{\gamma_r \alpha_r^n}{f'(\alpha_r)}. \tag{2.4}$$

3. The restricted period of a recurrence. Let p be a prime number which does not divide the constant term P_r of the characteristic polynomial (1.2). The least positive integer n such that the congruences

$$\alpha_1^n \equiv \alpha_2^n \equiv \ldots \equiv \alpha_r^n \pmod p \tag{3.1}$$

hold in the root field $\mathfrak{R}$ is called the *restricted period* of p in the recurrence (1.1) or the polynomial (1.2) **(3)**.

If ρ is the restricted period of p, (3.1) holds if and only if ρ divides n. Furthermore we have the congruence

$$W_{n+\rho} \equiv CW_n \pmod p, \qquad C \not\equiv 0 \pmod p, \tag{3.2}$$

where the residue C depends only on p and the recurrence (1.1). Consequently, p is a divisor of (W) if and only if it divides one of the ρ numbers

$$W_1, W_2, \ldots, W_{\rho-1}, W_\rho.$$

Now let (L) denote that particular recurring sequence with the initial values

$$L_1 = L_2 = \ldots = L_{r-2} = 0, \qquad L_{r-1} = 1.$$

For this sequence the polynomial $w(z)$ is one, so that all the γ_i are one, and by (2.4)

$$L_n = \frac{\alpha_1^n}{f'(\alpha_1)} + \ldots + \frac{\alpha_r^n}{f'(\alpha_r)}. \tag{3.3}$$

In case $r = 2$, L_n reduces to the well-known Lucas function

$$\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}.$$

We shall accordingly refer to (L) as the "Lucas sequence belonging to $f(z)$."

Every prime number p not dividing P_r is a maximal divisor of (L), and the first zero of order $r-1$ of p in (L) is simply the restricted period of $f(x)$ modulo p. We accordingly call ρ the *rank* of p in (L). Furthermore, every maximal divisor of (L) is regular.

4. Statement of results. Let $\Lambda(W)$ denote the rational integer

$$\Lambda(W) = DP_r\Delta(W). \tag{4.1}$$

Evidently $\Lambda(W)$ is not zero. Let p be any prime not dividing $\Lambda(W)$. Let (L) be the Lucas sequence belonging to $f(z)$, and let (M) be the Lucas sequence belonging to $g(z)$ of (2.3). Since p does not divide $\Lambda(W)$, it is a maximal prime divisor of both (L) and (M).

THEOREM 4.1. *Let p be a prime number not dividing $\Lambda(W)$ of (4.1). Then a necessary condition that p be a maximal divisor of (W) is that its rank in (M) divide its rank in (L).*

THEOREM 4.2. *The condition of Theorem* 4.1 *is sufficient for p to be a maximal prime divisor of* (W) *provided that $f(z)$ and p are restricted as follows:*

(i) *$f(z)$ is of odd degree and irreducible;*
(ii) *$p-1$ is prime to the degree r of $f(z)$;*
(iii) *$f(z)$ is irreducible modulo p.*

5. Proof of necessity of condition. We first prove Theorem 4.1. Let p be any prime not dividing $\Lambda(W)$, and assume that p is a maximal divisor of (W). Then there exists a positive integer k such that

$$W_k \equiv W_{k+1} \equiv \dots \equiv W_{k+r-2} \equiv 0 \pmod{p},$$

but

$$W_{k+r-1} \equiv C \not\equiv 0 \pmod{p}.$$

The sequence (T) defined by $T_n = W_{n+k} - CL_n$ satisfies the recurrence (1.1) and has its r initial values $T_0, \dots, T_{r-1}$ all divisible by p. Consequently, p divides every term of (T); in other words the congruences

$$W_{n+k} \equiv CL_n \pmod{p} \tag{5.1}$$

$$C \not\equiv 0 \pmod{p} \tag{5.2}$$

are necessary conditions for p to be maximal divisor of (W). For a fixed positive k and any rational integer C, they are also sufficient conditions for p to be maximal in (W); for since p does not divide P_r, it is maximal in (L).

Since p does not divide the discriminant D of $f(z)$, it is unramified in the root field $\mathfrak{R}$. Consequently its prime ideal factorization in $\mathfrak{R}$ is of the form

$$p = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_s \tag{5.3}$$

where the $\mathfrak{p}$ are distinct prime ideals.

Let ρ_j denote the restricted period of $f(z)$ modulo $\mathfrak{p}_j$; that is, ρ_j is the least positive integer n such that the congruences

$$\alpha_1^n \equiv \alpha_2^n \equiv \dots \equiv \alpha_r^n \pmod{\mathfrak{p}_j} \tag{5.4}$$

hold in $\mathfrak{R}$. Evidently the restricted period ρ of $f(z)$ modulo p is the least common multiple of the ρ_j.

If $f(z)$ is normal, its Galois group is transitive over the ideals $\mathfrak{p}_j$, and the Galois group is also transitive over the $\mathfrak{p}_j$ if $f(z)$ is irreducible modulo p. In either case, on applying the substitutions of the group to the congruences (5.4), we see that the ρ_j will all be equal. Hence we may state the following lemma:

LEMMA 5.1. *If $f(z) = 0$ is a normal equation or if $f(z)$ is irreducible modulo p, then with the notations of* (5.3)–(5.4), $\rho = \rho_j$ $(j = 1, 2, \dots, s)$.

Now let $\mathfrak{p}_j$ stand for any one of the prime ideal factors of p in the decomposition (5.3). Then the congruences (5.1) imply that for every n

$$W_{n+k} - CL_n \equiv 0 \pmod{\mathfrak{p}_j}, \qquad C \not\equiv 0 \pmod{\mathfrak{p}_j}. \tag{5.5}$$

On substituting for W_{n+k} and L_n from formulas (2.4) and (3.3) and then letting n range from 0 to $r-1$, we obtain r homogeneous linear congruences

$$\sum_{i=1}^{r} (\gamma_i\alpha_i^k - C)\frac{\alpha_i^n}{f'(\alpha_i)} \equiv 0 \pmod{\mathfrak{p}_j} \qquad (n = 0, 1, \ldots, r-1).$$

Now the algebraic numbers $\alpha_i^n f'(\alpha_i)^{-1}$ are integers modulo $\mathfrak{p}_j$, and the square of their determinant is D^{-1} which is both an integer mod $\mathfrak{p}_j$ and prime to $\mathfrak{p}_j$. Consequently

$$\gamma_1\alpha_1^k \equiv \gamma_2\alpha_2^k \equiv \ldots \equiv \gamma_r\alpha_r^k \equiv C \not\equiv 0 \pmod{\mathfrak{p}_j}. \tag{5.6}$$

Conversely these congruences imply the congruence (5.5). We may therefore state:

LEMMA 5.2. *If p does not divide the integer $\Lambda(W)$, then necessary and sufficient conditions that p should be a maximal divisor of the sequence (W) are that for some fixed positive integer k, the congruences (5.6) hold for every prime ideal factor $\mathfrak{p}_j$ of p in the root field of $f(z)$.*

Now let ρ_j be the restricted period of $f(x)$ modulo $\mathfrak{p}_j$ and σ_j the restricted period of $g(x)$ modulo $\mathfrak{p}_j$; that is, σ_j is the smallest positive value of n such that

$$\gamma_1^n \equiv \gamma_2^n \equiv \ldots \equiv \gamma_r^n \pmod{\mathfrak{p}_j}.$$

Then the restricted period σ of $g(x)$ modulo p is evidently the least common multiple of the σ_j.

On raising each term in (5.6) to the ρ_jth power, we see that σ_j must divide ρ_j. Hence σ must divide ρ, completing the proof.

6. Proof of sufficiency. It follows from the results of § 5 that if p does not divide $\Lambda(W)$, the conditions

$$\sigma_j \text{ divides } \rho_j \qquad (j = 1, 2, \ldots, s) \tag{6.1}$$

are necessary for the congruences (5.6) to hold. To answer the question of when these conditions are sufficient, we begin by studying the congruence

$$\gamma\alpha^k \equiv C \pmod{\mathfrak{p}}. \tag{6.2}$$

Here α as before is any root of $f(z)$, γ is an integer of the root field $\mathfrak{R}$ of $f(z)$, C is a rational integer, $\mathfrak{p}$ any prime ideal of $\mathfrak{R}$ dividing neither α nor γ, and k is a positive integer.

For brevity, we shall use the following special notations in this section. Since all congruences will be to the same modulus, we shall repress the mod $\mathfrak{p}$, writing (6.2) for example as $\gamma\alpha^k \equiv C$. $\gamma \equiv \text{Int}$ means there exists a rational integer g such that $\mathfrak{p}$ divides $\gamma - g$. Clearly

$$\gamma \equiv \text{Int} \quad \textit{if and only if } \gamma^{p-1} \equiv 1. \tag{6.3}$$

$\gamma \equiv Pr(\alpha)$ means γ is congruent modulo $\mathfrak{p}$ to a power of α. $ex(\gamma)$ means the exponent to which γ belongs modulo $\mathfrak{p}$; that is, the least positive value of n

such that $\gamma^n \equiv 1$. $rx(\gamma)$ means the restricted exponent of γ modulo $\mathfrak{p}$; that is, the least positive value of n such that $\gamma \equiv$ Int. Evidently

(6.4) $$\gamma^n \equiv \text{Int } \textit{if and only if } rx(\gamma) \textit{ divides } n.$$

Let

(6.5) $$\nu = ex(\gamma), \quad \sigma = rx(\gamma), \quad \gamma^\sigma \equiv g, \quad \delta = ex(g).$$

LEMMA 6.1. *With the notations of* (6.5),

(6.6) $$\nu = \sigma\delta$$

Proof: Evidently ν divides $\sigma\delta$. Let $(\nu, p-1) = t$ so that $\nu = \nu_0 t$ and $p-1 = lt$ with $(\nu_0, l) = 1$. Since $\gamma^{\nu_0(p-1)} \equiv 1$, $\gamma^{\nu_0} \equiv$ Int by (6.3). Consequently by (6.4), σ divides ν_0. Let $\nu_0 = \kappa\sigma$. Then

$$1 \equiv \gamma^\nu \equiv \gamma^{\nu_0 t} \equiv \gamma^{\kappa\sigma t} \equiv g^{\kappa t}.$$

Therefore $\delta | \kappa t$. Hence $\sigma\delta | \sigma\kappa t$, $\sigma\delta | \nu_0 t$ or $\sigma\delta$ divides ν. Hence $\sigma\delta = \nu$, completing the proof.

LEMMA 6.2. *If the irreducible congruence mod $\mathfrak{p}$ with rational integral coefficients of which γ is a root is of degree t, and if t is prime to $p-1$, where p is the rational prime corresponding to $\mathfrak{p}$, then the exponent ν to which γ belongs modulo $\mathfrak{p}$ is of the form* (6.6) *with σ and δ as before, but in addition σ, δ are coprime, σ divides $(p^t-1)/(p-1)$, δ divides $p-1$ and*

$$(\sigma, p-1) = 1.$$

Proof: Let the irreducible congruence be

$$z^t - R_1 z^{t-1} \dots + (-1)^t R_t \equiv 0 \qquad (\text{mod } \mathfrak{p})$$

where the R_t are rational integers. The roots of (6.6) are $\gamma, \gamma^p, \gamma^{p^2}, \dots \gamma^{p^{t-1}}$ Hence

$$\gamma^{\frac{p^t-1}{p-1}} \equiv R_t \equiv \text{Int}.$$

Therefore by (6.4), $\sigma | (p^t-1)/(p-1)$; obviously δ divides $p-1$. Now $((p^t-1)/(p-1), p-1) = (t, p-1) = 1$. Hence $(\sigma, \delta) = (\sigma, p-1) = 1$ which completes the proof.

Under the hypotheses of lemma 6.2 it is not difficult to show that δ is the exponent to which R_t in (6.8) belongs modulo p.

LEMMA 6.3. *With the hypotheses of Lemma* 6.2,

$$\gamma\alpha^k \equiv \text{Int } \textit{if and only if } \gamma^{p-1} \equiv Pr(\alpha).$$

Proof. If $\gamma\alpha^k \equiv$ Int, then

$$\gamma^{p-1}\alpha^{k(p-1)} \equiv 1$$

which implies $\gamma^{p-1} \equiv Pr(\alpha)$. Assume conversely that for some integer $l > 0$, $\gamma^{p-1} \equiv \alpha^l$.

Now $(\sigma, p-1) = 1$ by Lemma 6.2. Hence integers u and r exist such that $u\sigma + r(p-1) = 1$. Hence

$$\gamma = \gamma^{u\sigma + r(p-1)} \equiv g^u \alpha^{rl}.$$

Hence for some positive k, $\gamma\alpha^k \equiv \text{Int}$, completing the proof.

LEMMA 6.4. *If the restricted exponent σ of γ is prime to $p-1$ and divides the restricted exponent of α, then $\gamma^{p-1} \equiv Pr(\alpha)$.*

Proof. Let $\rho = rx(\alpha)$. Since $\gamma^{\sigma(p-1)} \equiv 1$, $ex(\gamma^{p-1})$ divides σ. Hence $ex(\gamma^{p-1})$ divides $rx(\alpha)$ or $ex(\gamma^{p-1})$ divides $ex(\alpha)$ by applying Lemma 6.1 to α instead of to γ. Hence $\gamma^{p-1} \equiv Pr(\alpha)$; for the multiplicative group of residues prime to $\mathfrak{p}$ is cyclic.

We may draw the following conclusion from the preceding lemmas which completes our investigation of the congruence (6.2).

LEMMA 6.5. *If the degree of γ modulo $\mathfrak{p}$ is prime to $p-1$, then a necessary and sufficient condition that the congruence* (6.2) *holds is that the restricted period of γ modulo $\mathfrak{p}$ divides the restricted period of γ modulo $\mathfrak{p}$.*

7. Proof of sufficiency concluded. We may now prove Theorem 4.2 as follows: Since $f(z)$ is irreducible modulo p, p does not divide P_r and p is unramified. Consequently its prime ideal factorization is as in (5.3). Let $\mathfrak{p}_j$ denote any prime ideal factor of p. By lemma 5.1, $\rho = \rho_j$ and $\sigma = \sigma_j$ and σ divides ρ by hypothesis. Also since $f(z)$ is irreducible modulo p, the degree t of γ is a divisor of r, so that t is prime to $p-1$. Consequently by Lemma 6.5,

$$\gamma\alpha^k \equiv C \not\equiv 0 \pmod{\mathfrak{p}_j}. \tag{7.1}$$

Here k may depend on j.

Now raise the congruence (7.1) successively to the $p, p^2, \ldots, p^{r-1}$ powers. Since $f(z)$ is irreducible mod p, its roots mod p and mod $\mathfrak{p}_j$ are the powers of any particular root α; that is, for a suitable numbering of the roots

$$\alpha_i \equiv \alpha^{p^{i-1}} \pmod{p} \qquad (i = 1, 2, \ldots, r).$$

Hence since $w(z)$ has rational integer coefficients,

$$\gamma^{p^{i-1}} \equiv w(\alpha^{p^{i-1}}) \equiv w(\alpha_i) \equiv \gamma_i \pmod{p}.$$

Therefore we obtain from (7.1) the congruences (5.6) and k is seen to be independent of j. But as was remarked in section 5, (5.6) implies congruences (5.1) and (5.2). Consequently p is a maximal divisor of (W), completing the proof.

8. Conclusion. A numerical example. Consider any integral recurrent sequence (W) defined by the recurrence $W_{n+3} = W_{n+2} + 4W_{n+1} + W_n$.

The characteristic polynomial of this recurrence $z^3 - z - 4z^2 - 1$ is irreducible and its discriminant is 169, a perfect square. Consequently, $f(z)$ is normal.

For every prime p congruent to 5 mod 6, $p-1$ is prime to $r=3$. Hence all the restrictive hypotheses of theorem 4.2 are met except possibly the irreducibility of $f(z)$ modulo p.

Consider the prime $p=5$. Then $f(z)$ is reducible modulo 5; in fact

$$f(z) \equiv (z-1)(z-2)(z-3) \qquad (\text{mod } 5).$$

Consequently the restricted period of $f(z)$ modulo 5 (that is, the rank of 5 in (L)) is four. Since $g(z)$ is evidently completely reducible modulo 5, the rank of 5 in (M) always divides the rank of 5 in (L).

Now suppose the initial values of (W) are chosen so that five does not divide $\Delta(W)$ of (4.1), which amounts to saying that the recurrence (W) is of order three modulo five. Then five may or may not be a maximal divisor of (W). For example, if $W_0=1$, $W_1=1$, $W_2=0$ then $\Delta(W)=5239$. But $W_3=5$ and p is maximal. If $W_0=1$, $W_1=3$, $W_2=5$ then $\Delta(W)=12337$. But $W_3=18$ and (W) has period four modulo 5. Hence p is not maximal in this recurrence.

To illustrate the possibility of an irregular maximal prime divisor, consider the recurrence $W_{n+3}=7W_{n+2}+36W_{n+1}+29W_n$ with $W_0=7$, $W_1=7$, and $W_2=1$. Then if we take $p=7$, p is obviously maximal in (W). But p is irregular. For on computing the first nineteen terms of (W) mod 49, we obtain

$$7, 7, 1, 21, 43, 8, 8, 23, 44, 45, 18, 33, 28, 44, 19, 30, 14, 14, 2.$$

Since the last three terms are double the first three,

$$W_{n+16} \equiv 2W_n \qquad (\text{mod } 49)$$

so that no term of (W) is divisible by 7^2.

There exist for cubic sequences fairly simple criteria distinguishing regular and irregular primes. These I plan to give elsewhere.

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ON DISCRIMINANTS OF BINARY QUADRATIC FORMS WITH A SINGLE CLASS IN EACH GENUS

S. CHOWLA AND W. E. BRIGGS

1. Introduction. Consider the classes of positive, primitive binary quadratic forms $ax^2 + bxy + cy^2$ of discriminant $-\Delta = d = b^2 - 4ac < 0$. Dickson (2, p. 89) lists 101 values of Δ such that $-\Delta$ is a discriminant having a single class in each genus. The largest value given is 7392, and Swift (7) has shown that there are no more up to 10^7. Sixty-five of these values are divisible by 4. For these values, $\Delta/4$ is called an idoneal number; its properties were investigated by Euler.

We write as usual

$$L_k(s) = \sum_1^\infty \chi(n)n^{-s}, \qquad \Re(s) > 0,$$

where throughout $\chi(n)$ is a real non-principal character modulo k; also $\zeta(s)$ is the Riemann zeta function defined for $R(s) > 1$ by

$$\zeta(s) = \sum_1^\infty n^{-s}.$$

We prove the two theorems:

THEOREM I. *If $\Delta > 10^{60}$, there is at most one fundamental discriminant $-\Delta$ with a single class in each genus.*

THEOREM II. *If $L_k(53/54) > 0$ for $k > 10^{14}$, there are for $\Delta > 10^{14}$ no fundamental discriminants $-\Delta$ with a single class in each genus.*

Chowla (1) proved that as d approaches $-\infty$, the number of classes in each genus tends to ∞, so that after some indeterminate point, there are no discriminants with a single class in each genus. This also follows from the well-known inequality of Siegel (6) which states that $L_k(1) > k^{-\epsilon}$, $k > k_0(\epsilon)$.

If $h(d)$ is the class number, then for fundamental discriminants $d < -4$,

$$h(d) = \frac{\sqrt{\Delta}}{\pi}\sum_1^\infty \left(\frac{d}{n}\right)\frac{1}{n} = \frac{\sqrt{\Delta}}{\pi} L_\Delta(1),$$

since the Kronecker symbol is a real non-principal character modulo Δ. The number of genera into which these classes are divided is either 2^{t-1} or 2^t, where t is the number of distinct prime factors of d.

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2. Some lemmas.

LEMMA 1. $|\zeta(\frac{1}{2} + it)| < 2(|t| + 1)$.

Since (**8**, p. 14)

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{x - [x]}{x^{s+1}}\,dx, \qquad \Re(s) > 0,$$

$$|\zeta(\tfrac{1}{2} + it)| < \left|\frac{\frac{1}{2} + it}{-\frac{1}{2} + it}\right| + (\tfrac{1}{2} + |t|)\int_1^\infty x^{-3/2}dx = 1 + 2(\tfrac{1}{2} + |t|).$$

LEMMA 2. $|L_k(\frac{1}{2} + it)| < (2|t| + 1)\sqrt{k}\log k$.

Let

$$S(x) = \sum_{n<x} \chi(n).$$

Then, for $\Re(s) > 0$,

$$L_k(s) = \sum_1^\infty \frac{S(n) - S(n-1)}{n^s} = \sum_1^\infty S(n)\{n^{-s} - (n+1)^{-s}\}$$

$$= \sum_1^\infty S(n)\, s\int_n^{n+1} \frac{dx}{x^{s+1}} = s\int_1^\infty \frac{S(x)}{x^{s+1}}\,dx.$$

But $|S(x)| < \sqrt{k}\log k$ (**5**, Satz 494), hence,

$$|L_k(\tfrac{1}{2} + it)| < (\tfrac{1}{2} + |t|)\sqrt{k}\log k\int_1^\infty x^{-3/2}\,dx = 2\sqrt{k}(\tfrac{1}{2} + |t|)\log k.$$

We define for complex $s \neq 1$,

$$F(s) = \zeta(s)\,L_k(s).$$

For $\Re(s) > 1$, we write

$$\zeta(s)L_k(s) = \sum_1^\infty a_n n^{-s}.$$

Then $a_1 = 1$, $a_n \geq 0$, and $a_n \geq 1$ if $n = r^2$ (**3**, p. 428). Let

$$G(x) = \sum_1^\infty a_n e^{-nx}, \qquad x > 0.$$

By a theorem of Mellin (**5**, Satz 231),

$$e^{-x} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)x^{-s}ds, \qquad x > 0.$$

Therefore

$$G(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)x^{-s}F(s)\,ds.$$

This integral can be evaluated by applying Cauchy's Theorem to the rectangle with vertices $2 \pm Ti$, $\frac{1}{2} \pm Ti$, $T > 0$. On the horizontal paths, the integral has the order (**5**, Satz 229, Satz 407)

$$O\left(\frac{T^{5/2}}{e^{\frac{1}{2}\pi T}\sqrt{x}}\right).$$

Letting $T \to \infty$, we obtain, because of the singularity at $s = 1$,

$$G(x) = \frac{L_k(1)}{x} + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)x^{-s}F(s)\,ds. \tag{1}$$

LEMMA 3.

$$|\Gamma(\tfrac{1}{2} + it)| = \frac{\sqrt{\pi}}{\sqrt{\cosh \pi t}}.$$

This follows from $\Gamma(s)\,\Gamma(1 - s) = \pi/\sin s\pi$.
From Lemma 3,

$$|\Gamma(\tfrac{1}{2} + it)| < \sqrt{2\pi}\,e^{-\frac{1}{2}\pi|t|}. \tag{2}$$

LEMMA 4. *If $L_k(53/54) \geq 0$ for $k > 10^{14}$, then*

$$L_k(1) > \frac{1}{54}k^{1/27}.$$

From (1), (2), Lemmas 1 and 2, we obtain

$$\left|G(x) - \frac{L_k(1)}{x}\right| < \frac{1}{2\pi\sqrt{x}}\int_{-\infty}^{\infty} \sqrt{2\pi}\,e^{-\frac{1}{2}\pi|t|}2\sqrt{k}(2|t|^2 + 3|t| + 1)\log k\,dt$$

$$= \frac{2\sqrt{2k}\log k}{\sqrt{\pi x}}\int_0^{\infty}(2t^2 + 3t + 1)\,e^{-\frac{1}{2}\pi t}\,dt$$

$$= \frac{2\sqrt{2k}\log k}{\sqrt{\pi x}}\left(\frac{32}{\pi^3} + \frac{12}{\pi^2} + \frac{2}{\pi}\right) < \frac{5\sqrt{k}\log k}{\sqrt{x}},$$

and

$$\left|G\left(\frac{x}{k}\right) - \frac{kL_k(1)}{x}\right| < \frac{5k\log k}{\sqrt{x}}. \tag{3}$$

Next for $\Re(s) > 1$,

$$\Gamma(s) = \int_0^{\infty} e^{-x}x^{s-1}\,dx = \left(\frac{n}{k}\right)^s \int_0^{\infty} e^{-nx/k}x^{s-1}\,dx,$$

$$k^s\Gamma(s)\,F(s) = \int_0^{\infty} x^{s-1}G\left(\frac{x}{k}\right)dx.$$

Therefore

$$k^s\Gamma(s)F(s) - \frac{kL_k(1)t^{s-1}}{s-1} = \int_0^t x^{s-1}\left\{G\left(\frac{x}{k}\right) - \frac{kL_k(1)}{x}\right\}dx + \int_t^{\infty} x^{s-1}G\left(\frac{x}{k}\right)dx \tag{4}$$

$$= I_1 + I_2.$$

From now on suppose $53/54 \leq s < 1$. Then (4) still holds on noting (3).

Now set $t = 1/k$. Then, for $k > 10^{14}$,

$$I_2 = \int_{k^{-1}}^{\infty} x^{s-1} G\left(\frac{x}{k}\right) dx = k^s \int_{k^{-2}}^{\infty} x^{s-1} G(x)\, dx > k^s \int_{k^{-2}}^{1} G(x)\, dx$$

$$> k^s \int_{k^{-2}}^{1} \sum_{r=1}^{\infty} e^{-r^2 x}\, dx > k^s \sum_{r=1}^{100} \frac{1}{r^2} (e^{-r^2/k^2} - e^{-r^2})$$

$$> k^s \left\{ \left[\frac{\pi^2}{6} - 10^{-2}\right][1 - 10^{-24}] - \sum_{1}^{3} \frac{e^{-r^2}}{r^2} - 97\frac{e^{-16}}{16} \right\}$$

$$> \frac{5}{4} k^s.$$

$$|I_1| < \int_0^t x^{s-1} \frac{5k \log k}{\sqrt{x}} dx = \frac{5kt^{s-\frac{1}{2}}}{s - \frac{1}{2}} \log k.$$

Hence

$$I_1 + I_2 > \frac{5}{4} k^s - \frac{5k^{(3/2)-s}}{s - \frac{1}{2}} \log k;$$

and it is easily seen that for $k > 10^{14}$

$$I_1 + I_2 > k^s.$$

To complete the proof of Lemma 4, take $s = 53/54$. Then $\zeta(s) < 0$ and $L_k(s) > 0$. Hence the first term of (4) is non-positive and so

$$\frac{kL_k(1)t^{s-1}}{1 - s} > k^s$$

or

$$L_k(1) > (1 - s)k^{2(s-1)}. \tag{5}$$

This is the result, since (5) holds at 53/54.

Lemma 5. *If* $-d = \Delta > 10^{14}$ *then* $2^t < \Delta^{0.3}$, *and if* $-d = \Delta > 10^{60}$ *then* $2^t < \Delta^{0.2}$.

The smallest positive integer with r prime factors is the product of the first r primes. Let this product be P_r. Then the lemma follows easily by induction, since if

$$2^r < (P_r)^m,$$

$$2^{r+1} < 2(P_r)^m < (P_{r+1})^m, \qquad p_{r+1} > 2^{1/m},$$

and $r = 13$ is the smallest value of r such that $P_r > 10^{14}$, and $r = 37$ is the smallest value of r such that $P_r > 10^{60}$.

3. Proof of the theorems. We first prove Theorem II. From

$$h(d) = \frac{\sqrt{\Delta}}{\pi} L_\Delta(1),$$

and Lemma 4, we have for $\Delta > 10^{14}$,

$$h(d) > \frac{\sqrt{\Delta}}{\pi}\frac{1}{54\,\Delta^{1/27}} = \frac{\Delta^{25/54}}{54\pi}.$$

By Lemma 5, the number of genera is less than $\Delta^{0.3}$ for $\Delta > 10^{14}$. Therefore the theorem is true whenever

$$\frac{\Delta^{25/54}}{54\pi} > \Delta^{0.3}$$

which holds for $\Delta > 10^{14}$.

We now prove Theorem I. We assume there are two such discriminants d_1, d_2 with $\Delta_1 > \Delta_2 > 10^{60}$ and show that this leads to a contradiction. The tests given by Swift (7) for a discriminant to have more than a single class in each genus show that if d has a single class in each genus, then d, $d/4$, or $d/16$ is a fundamental discriminant. From this Theorem 1 can be extended to all discriminants without difficulty but with tedium.

Landau (4, p. 281) proved

$$\text{(6)} \qquad \frac{h(d_1)}{\sqrt{\Delta_1}\log^2\Delta_1} + \frac{h(d_2)}{\sqrt{\Delta_2}\log^2\Delta_2} > \frac{1}{5\log^5(\Delta_1\Delta_2)}.$$

By assumption

$$\text{(7)} \qquad h(d_1) < 2^{t_1} < \Delta_1^{\delta};\ h(d_2) < 2^{t_2} < \Delta_2^{\delta}, \qquad \delta < 1/5,$$

where the upper bound for δ follows from Lemma 4. From (6)

$$\frac{2}{\Delta_1^{\frac{1}{2}-\delta}\log^2\Delta_1} > \frac{1}{5\log^5(\Delta_2^2)} = \frac{1}{160\log^5\Delta_2},$$

or

$$\text{(8)} \qquad \log\Delta_2 > \Delta_1^{(1-2\delta)/10}.$$

Next define

$$P(s) = \zeta(s)\,L_{k_1}(s)\,L_{k_2}(s)\,L_{k_1\cdot k_2}(s),$$

where χ_1, χ_2, the characters in

$$L_{k_1}(s),\ L_{k_2}(s),$$

are real primitive non-principal characters modulo k_1 and k_2, $k_1 \neq k_2$. Also

$$L_{k_1\cdot k_2}(s) = \sum_1^{\infty}\frac{\chi_1(n)\,\chi_2(n)}{n^s}.$$

Write for $\Re(s) > 1$

$$P(s) = \sum_1^{\infty} b_n n^{-s}.$$

Again $b_1 = 1$, $b_n \geqslant 0$, and $b_n \geqslant 1$ if $n = r^2$. Let

$$H(x) = \sum_1^{\infty} b_n e^{-nx}, \qquad x > 0.$$

As we obtained (1), we now obtain

$$H(x) = \frac{L^*}{x} + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)\, x^{-s} P(s)\, ds, \tag{9}$$

where

$$L^* = L_{k_1}(1)\, L_{k_2}(1)\, L_{k_1,k_2}(1).$$

From (9), (2), Lemmas 1 and 2, results,

$$\left|H(x) - \frac{L^*}{x}\right|$$

$$< \frac{1}{2\pi\sqrt{x}}\int_{-\infty}^{\infty} \sqrt{2\pi}\, e^{-\frac{1}{2}\pi|t|} 2(|t|+1)(2|t|+1)^3 k_1k_2 \log k_1 \log k_2 \log (k_1k_2)\, dt$$

$$= \frac{2\sqrt{2}\, k_1k_2 \log k_1 \log k_2 \log(k_1k_2)}{\sqrt{\pi x}}\int_0^{\infty} e^{-\frac{1}{2}\pi t}(8t^4 + 20t^3 + 18t^2 + 7t + 1)\, dt$$

$$= \frac{2\sqrt{2}\, k_1k_2 \log k_1 \log k_2 \log(k_1k_2)}{\sqrt{\pi x}}\left(\frac{6144}{\pi^5} + \frac{1920}{\pi^4} + \frac{288}{\pi^3} + \frac{28}{\pi^2} + \frac{2}{\pi}\right)$$

$$< \frac{100\, k_1k_2 \log k_1 \log k_2 \log (k_1k_2)}{\sqrt{x}}.$$

Therefore

$$\left|H\left(\frac{x}{k_1k_2}\right) - \frac{L^* k_1k_2}{x}\right| < \frac{100(k_1k_2)^{3/2} \log k_1 \log k_2 \log(k_1k_2)}{\sqrt{x}}. \tag{10}$$

As we obtained (4), we now obtain, for $\Re(s) > 1$,

$$(k_1k_2)^s\Gamma(s)\, P(s) - \frac{k_1k_2L^* q^{s-1}}{s-1} \tag{11}$$

$$= \int_0^q x^{s-1}\left\{H\left(\frac{x}{k_1k_2}\right) - \frac{k_1k_2L^*}{x}\right\} dx + \int_q^{\infty} x^{s-1} H\left(\frac{x}{k_1k_2}\right) dx$$

$$= J_1 + J_2.$$

Suppose now $53/54 < s < 1$. Then (11) still holds by (10). Put $q = (k_1k_2)^{-2}$. As before, we obtain

$$J_2 > \frac{5}{4}(k_1k_2)^s,$$

and

$$|J_1| < 100(k_1k_2)^{3/2} \log k_1 \log k_2 \log(k_1k_2)\, \frac{q^{s-\frac{1}{2}}}{s-\frac{1}{2}}.$$

Hence for $s > 53/54$

$$J_1 + J_2 > (k_1k_2)^s, \qquad k_1, k_2 > 10^{60}.$$

Therefore from (11) follows

LEMMA 6. *If* $P(s_0) < 0$, $53/54 < s_0 < 1$, *then*

$$L_{k_1}(1)\, L_{k_2}(1)\, L_{k_1, k_2}(1) > (1 - s_0)(k_1 k_2)^{3(s_0 - 1)}$$

for $k_1, k_2 > 10^{60}$.

From (7),

(12) $$L_{\Delta_1}(1) < \frac{\pi}{\Delta_1^{1-s}}\,;\; L_{\Delta_2}(1) < \frac{\pi}{\Delta_2^{1-s}}\,.$$

But

$$\frac{\pi}{\Delta_1^{1-s}} < \frac{1}{54\Delta_1^{1/27}}, \qquad \Delta_1 > 10^{60},$$

and therefore by Lemma 4,

$$L_{\Delta_1}(53/54) < 0,$$

which means that

$$L_{\Delta_1}(s_0) = 0, \qquad 53/54 < s_0 < 1,$$

and that $P(s_0) = 0$. Furthermore

(13) $$L_{\Delta_1}(1) = (1 - s_0)\, L'_{\Delta_1}(v), \qquad s_0 < v < 1.$$

Let $53/54 < s < 1$ and $S(x) = \sum_1^x \chi(n)$. Then

$$L'_k(s) = -\sum_1^\infty \frac{\chi(n) \log n}{n^s} = -\sum_{x=1}^\infty S(x)\left[\frac{\log x}{x^s} - \frac{\log(x+1)}{(x+1)^s}\right],$$

so that

$$\begin{aligned}
|L'_k(s)| &< \sum_{x=1}^k x\left|\frac{\log x}{x^s} - \frac{\log(x+1)}{(x+1)^s}\right| + \sum_{k+1}^\infty \sqrt{k}\log k\left|\frac{\log x}{x^s} - \frac{\log(x+1)}{(x+1)^s}\right| \\
&< \sum_{x=1}^k x\left|\frac{1 - s\log(x + c_x)}{(x + c_x)^{s+1}}\right| + \frac{\log^2 k}{k^{s-\frac{1}{2}}}, \qquad 0 < c_x < 1, \\
&< 1 + 1 + \sum_{x=3}^k x\,\frac{s\log x}{x^{s+1}} + \frac{\log^2 k}{k^{s-\frac{1}{2}}} \\
&< 2 + 54\log k[k^{1/54} - 2^{1/54}] + 10^{-24}\log^2 k, \qquad k > 10^{60}. \\
&< 55\, k^{1/54}\log k.
\end{aligned}$$

Also

$$L_{\Delta_1}(1) = \frac{\pi}{\sqrt{\Delta_1}}\, h(d_1) \geqslant \frac{\pi}{\sqrt{\Delta_1}}\,.$$

Therefore from (13), we obtain

(14) $$1 - s_0 > \frac{\pi}{55\Delta_1^{14/27}\log\Delta_1}\,.$$

By (8),

$$\Delta_2 > \exp \Delta_1^{3/50} > \Delta_1^5, \qquad \Delta_1 > 10^{60},$$

or

(15) $$\Delta_1 < \Delta_2^{1/5}.$$

As is well known (**4**, p. 281),

$$L_{\Delta_1,\Delta_2}(1) < 3\log(\Delta_1\Delta_2).$$

Applying this, (12), (14), (15) to Lemma 6, gives

$$L_{\Delta_2}(1) > \frac{(\Delta_2^{6/5})^{-1/18}}{165(\Delta_2^{1/5})^{\delta+1/54}\log(\Delta_2^{1/5})\log(\Delta_2^{6/5})}$$

$$> \frac{1}{40\Delta_2^{(2/27)+\delta/5}\log^2\Delta_2} > \frac{1}{40\Delta_2^{0\cdot 2}} > \frac{\pi}{\Delta_2^{1-\delta}},$$

which contradicts (12).

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ON INTEGRAL CLOSURE

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1. Introduction. Let J be an integral domain (i.e., a commutative ring without divisors of zero) with unit element, F its quotient field and $J[x]$ the integral domain of polynomials with coefficients from J. The domain J is called integrally closed if every root of a monic polynomial over J which is in F also is in J. If J has unique factorization into primes, a well-known lemma of Gauss asserts: "If $p(x)$ is a polynomial in $J[x]$ factoring over F, then $p(x)$ factors over J." For proof see (**2**, p. 73). We shall show that if J is integrally closed but unique factorization is not assumed in J and if $p(x) = ax^m + \ldots + a_m$ is in $J[x]$ and $p(x) = g(x)\,h(x)$ in $F[x]$, then $ap(x)$ factors in $J[x]$. The case $a = 1$, which asserts that the Gauss lemma holds for monic polynomials, is important in many applications.

We show further a hereditary property of integral closure, namely, that $J[x]$ is integrally closed if J is integrally closed. These two theorems permit us to generalize a theorem on the relation between the Galois group of a monic polynomial over J and the Galois group of the corresponding polynomial mod p where p is a prime ideal of J.

2. Theorems on integral domains. An element β algebraic over F is called an algebraic integer if β satisfies a monic equation (not necessarily irreducible) with coefficients in J. A well-known theorem on symmetric polynomials then shows that the algebraic integers form a ring J^* and that this ring is integrally closed. Moreover if J is integrally closed and if an algebraic integer β lies in F, then it must lie in J. From our definition, it follows that the conjugates over F of an algebraic integer are also integral, and so the monic irreducible equation over F of an integer has its coefficients in J.

THEOREM 1. *Let J be an integrally closed integral domain with unit element, F its quotient field. Let $f(x) \in J[x]$ and $f(x) = g(x)\,h(x)$ where $g(x), h(x) \in F[x]$. Let $f(x), g(x), h(x)$ have first coefficients a, b, c respectively. Then*

$$\frac{a}{b}\,g(x), \quad \frac{a}{c}\,h(x)$$

have integral coefficients. Hence

$$af(x) = \left(\frac{a}{b}\,g(x)\right)\left(\frac{a}{c}\,h(x)\right)$$

is a decomposition of $af(x)$ in $J[x]$.

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Proof. Let ρ be a root of $f(x)$. An argument completely analogous to that given in (**1**, p. 91) for the case that J is the domain of algebraic integers in the usual sense shows that

$$\frac{f(x)}{x-\rho}$$

has integral coefficients. Applying this to all the roots ρ of $h(x)$, we deduce that

$$\frac{cf(x)}{h(x)} = cg(x) = \frac{a}{b}g(x)$$

has integral coefficients. For $a = 1$ we have:

COROLLARY. *If J is integrally closed and the monic polynomial $f(x) \in J[x]$ factors in $F[x]$, then it also factors in $J[x]$.*

For the applications of Theorem 1 and its Corollary, it will be necessary to show that the property of algebraic closure carries over to the polynomial domain $J[x]$.

THEOREM 2. *If J is integrally closed, then $J[x]$ is integrally closed.*

Let $f(x)/g(x)$ be a root of a monic polynomial with coefficients in $J[x]$. Since unique factorization holds in $F[x]$, it follows that $F[x]$ is integrally closed. Hence $g(x)$ must be an element of F and we can choose it in J. Let now $f(x)/\alpha$, $f(x) \in J[x]$, $\alpha \in J$ satisfy a monic equation with coefficients in $J[x]$. Since the domain of integers over J is integrally closed, $f(\beta)/\alpha$ must be integral for all integers β. Let

$$f(x) = A_0x^m + \ldots,$$

then

$$\frac{f(x)-f(\beta)}{\alpha} = \frac{(x-\beta)f_1(x)}{\alpha}$$

is integral valued for all integral values of x. Moreover the first coefficient of $f_1(x)$ is A_0. Suppose now that we have constructed a polynomial:

$$\phi_s(x) = \frac{(x-\rho_1)\ldots(x-\rho_s)f_s(x)}{\alpha},$$

where the ρ_i are integers such that $\phi_s(x)$ is integral, whenever x is integral and such that the first coefficient of $f_s(x)$ is A_0. Let ρ_{s+1} be a root of the equation

$$(x-\rho_1)\ldots(x-\rho_s) = 1.$$

Then ρ_{s+1} is an integer and $\phi_s(\rho_{s+1}) = f_s(\rho_{s+1})/\alpha$. Hence

$$\frac{(x-\rho_1)\ldots(x-\rho_s)f_s(x)}{\alpha} - \frac{(x-\rho_1)\ldots(x-\rho_s)f(\rho_{s+1})}{\alpha} = \frac{(x-\rho_1)\ldots(x-\rho_{s+1})f_{s+1}(x)}{\alpha}$$

is integral whenever x is integral and $f_{s+1}(x)$ has again A_0 as first coefficient. Continuing in this manner, we arrive at a polynomial

$$\frac{A_0(x-\rho_1)\ldots(x-\rho_m)}{\alpha}$$

which is integral whenever x is an integer. Let β be a root of the equation,

$$(x-\rho_1)\ldots(x-\rho_m)=1.$$

Then β is an integer and it follows that A_0 is divisible by α. We may therefore write:

$$\frac{F(x)}{\alpha}=bx^m+\frac{g(x)}{\alpha}, \qquad b\in J,\ g(x)\in J[x],$$

where $g(x)$ is a polynomial of degree at most $m-1$. Substituting in the equation for $F(x)/\alpha$, we see that $g(x)/\alpha$ is also root of a monic polymonial with coefficients in $J[x]$. Theorem 2 now follows by induction.

COROLLARY. *If J is integrally closed, then $J[x_1, \ldots, x_n]$ is integrally closed.*

3. Application to Galois theory. The corollary can be used to generalize a theorem that has been known to hold for unique factorization domains (**2**, p. 190) as well as for algebraic number fields (**3**, p. 122, eq. 10.6).

THEOREM 3. *Let J be an integrally closed integral domain, p a prime ideal in J. Let $\bar{J}$ be the residue ring of J (mod p) and $f(x)$ a monic polynomial in $J(x)$, $\bar{f}(x)$ the corresponding polynomial in $\bar{J}(x)$. Let Δ, $\bar{\Delta}$, be the quotient fields of J and $\bar{J}$ respectively. If $f(x)$ and $\bar{f}(x)$ do not have any double roots, then the roots of $f(x)$ and $\bar{f}(x)$ can be so numbered that the Galois group of $\bar{f}(x)$ is a subgroup of the Galois group of $f(x)$.*

A study of the proof of this theorem in (**2**, p. 190), readily shows that the assumption of unique factorization in J made there is used only to establish the factorization of a monic polynomial over the ring $J[u_1, \ldots, u_n]$ from its factorization in the quotient field of $J[u_1, \ldots, u_n]$. It can therefore be replaced by Theorem 1 coupled with the Corollary to Theorem 2. The proof itself is word by word the same as in (**2**).

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ON AN EXCEPTIONAL PHENOMENON IN CERTAIN QUADRATIC EXTENSIONS

H. B. MANN

Let Ω be a cyclic extension of degree l over the field Σ. Correcting an error which for some time had been haunting the literature, Hasse (**1**, p. 272) noted that for $l = 2$, the field Ω may contain a unit ξ such that

$$\xi^{l^\beta} \in \Sigma, \; \xi^{l^{\beta-1}} \notin \Sigma, \; \beta > 1.$$

Hasse also gave the example $\Sigma = \mathfrak{R}(\sqrt{-2})$, $\Omega = \Sigma(\sqrt{-1})$, where $\mathfrak{R}$ is the rational field and $\Omega \ni \sqrt[4]{-1}$. In this note, we shall give necessary and sufficient conditions under which this exceptional case arises.

THEOREM 1. *Let Ω be any field separable and cyclic of degree l (a prime) over a field Σ. There exists an element $\omega \in \Omega$ such that $\omega^{l^\beta} \in \Sigma$, $\omega^{l^{\beta-1}} \notin \Sigma$, $\beta > 2$, if and only if*

$$\text{(i)} \quad l = 2,$$

$$\text{(ii)} \quad \Omega = \Sigma(\sqrt{-1}),$$

$$\text{(iii)} \quad \Sigma \ni \theta + \theta^{-1},$$

where θ is a primitive 2^βth root of unity. Moreover

(iv) *Ω contains the 2^βth roots of unity,*

(v) $\omega = \alpha(1 + \theta), \quad \alpha \in \Sigma.$

Proof. Since Ω is cyclic, hence normal, over Σ and since

$$\Omega = \Sigma(\sqrt[l]{\omega^{l^\beta}}),$$

it is clear that the lth roots of unity must be in Σ. If l is odd, then

$$\omega^{l^\beta} = N(\omega^{l^{\beta-1}}) = N(\omega^{l^{\beta-2}})^l,$$

hence

$$\omega^{l^{\beta-1}} \in \Sigma$$

contrary to hypothesis. (N denotes the relative norm in Ω over Σ.) Hence $l = 2$. We then have

$$-\omega^{2^\beta} = N(\omega^{2^{\beta-1}}) = N(\omega^{2^{\beta-2}})^2$$

which shows that $\sqrt{-1} \notin \Sigma$ and $\Omega = \Sigma(\sqrt{-1})$. Furthermore, we must have $\omega^S = \theta\omega$, where S is the generating automorphism of Ω over Σ and θ a 2^βth

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root of unity. Moreover, θ cannot be a $2^{\beta-1}$th root of unity, otherwise we should have

$$(\omega^{2^{\beta-1}})^S = \omega^{2^{\beta-1}} \in \Sigma.$$

The equation $\omega^S = \theta\omega$ shows $N(\theta) = 1$. Hence $\theta^S = \theta^{-1}$, so that $\theta + \theta^{-1} \in \Sigma$ and $(1 + \theta^{-1})^S = \theta(1 + \theta^{-1})$. This gives

$$\left(\frac{\omega}{1+\theta^{-1}}\right)^S = \frac{\omega}{1+\theta^{-1}},$$

and shows that

$$\omega = \alpha(1 + \theta^{-1}), \quad \alpha \in \Sigma.$$

On the other hand, let the conditions (i), (ii), and (iii) be satisfied. Since $\theta^2 - \theta(\theta + \theta^{-1}) + 1 = 0$ and since $\Sigma(\theta) \ni \sqrt{-1}$, it follows that $\Omega \ni \theta$ and $\theta^S = \theta^{-1}$. Therefore

$$((1+\theta)^{2^{\beta-1}})^S = -(1+\theta)^{2^{\beta-1}} \notin \Sigma,$$
$$((1+\theta)^{2^\beta})^S = \quad (1+\theta)^{2^\beta} \in \Sigma.$$

This completes the proof of Theorem 1.

The condition (v) shows that β is bounded if Ω is a finite extension of $\mathfrak{R}$. We thus have

COROLLARY 1.1. *If Ω is a finite extension of the rationals, then β is bounded. If β is the largest value such that there exists a number ω in Ω for which $\omega^{2^\beta} \in \Sigma$, $\omega^{2^{\beta-1}} \notin \Sigma$, then $\omega \neq \alpha\omega_1{}^{1-S}$, $\alpha \in \Sigma$, $\omega_1 \in \Omega$.*

Otherwise $\omega = \alpha\omega_1{}^{1-S} = \alpha\omega_1{}^{1+S}\,\omega_1{}^{-2S} = \alpha^*\,\omega_1{}^{*2}$. But this shows $\omega_1{}^{*2^\beta} \notin \Sigma$, $\omega_1{}^{*2^{\beta+1}} \in \Sigma$ contrary to the significance of β.

The same argument also shows

COROLLARY 1.2. *If under the conditions of corollary 1, β is the largest value such that there is a unit $H \in \Omega$ for which $H^{2^\beta} \in \Sigma$, $H^{2^{\beta-1}} \notin \Sigma$, then H is not of the form $H_1{}^{1-S}\,\epsilon$, where H_1 is a unit of Ω, ϵ a unit of Σ.*

THEOREM 2. *The number ω in Theorem 1 can (under the conditions of Corollary 1) be chosen as a unit if and only if the ideal (2) is, in Σ, the $2^{\beta-1}$th power of a principal ideal (α).*

Proof. We have

$$(2) = (1+\theta)^{2^{\beta-1}}.$$

If $(2) = (\alpha^{2^{\beta-1}})$, then $(1 + \theta)/\alpha = \omega$ is a unit. On the other hand, if ω is a unit, then by Theorem 1.

$$\omega = \alpha(1 + \theta), \quad \alpha \in \Sigma.$$

Hence

$$(\alpha^{-1}) = (1+\theta),\ (2) = (\alpha^{-1})^{2^{\beta-1}}.$$

If β is chosen maximal, then the $2^{\beta+1}$th roots of unity are in Ω if and only if $\Sigma \ni \theta_1 - \theta_1{}^{-1}$, where θ_1 is a primitive $2^{\beta+1}$th root of unity. In this case, it is

trivial that ω can be chosen as a unit. A less trivial example is $\Sigma = \Re(\sqrt{7})$, $\Omega = \Re(\sqrt{7}, i)$, where the unit

$$H = \frac{1+i}{3+\sqrt{7}}$$

has the property $H^2 \notin \Sigma$, $H^4 \in \Sigma$.

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SOME RELATIONS BETWEEN VARIOUS TYPES OF NORMALITY OF NUMBERS

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1. Introduction. In this paper certain relations will be proved between ϵ-normality of integers, (k, ϵ)-normality of integers, and normality of real numbers. Also a new type of normality of numbers will be introduced, namely, quasi-normality, as defined below.

DEFINITION 1.1. A simply normal number *is a real number, expressed in some scale B, in which each digit of the scale B occurs with the asymptotic frequency* $1/B$.

DEFINITION 1.2. A normal number *is a real number, expressed in some scale B, in which every sequence of k digits of the scale B,* $k = 1, 2, 3, \ldots$, *occurs with the asymptotic frequency* $1/B^k$.

DEFINITION 1.3. *An integer*

$$m = a_{\mu-1}a_{\mu-2}\ldots a_1a_0 \qquad (a_{\mu-1} \neq 0),$$

where the a_i *are digits of some scale B, is* (k, ϵ)-normal *in the scale B for a given k and a given* $\epsilon > 0$, *if for every k-digit sequence* $b_1b_2\ldots b_k$,

$$\left|\frac{N(m, b_1b_2\ldots b_k)}{\mu - k + 1} - \frac{1}{B^k}\right| < \epsilon,$$

where $N(m, b_1b_2\ldots b_k)$ *is the number of occurrences of* $b_1b_2\ldots b_k$ *in m. For* $k = 1$ *we shall say simply that m is* ϵ-normal *in the scale B.*

DEFINITION 1.4. *A real number y, expressed in some scale B, is* quasi-normal *in the scale B if every number derived from y by selecting those digits whose positions in y form an arithmetic progression is a simply normal number.*

Definition 1.2 was originally given by Borel **(2)** as the characteristic property of normal numbers. Borel actually defined a number x to be normal in the scale B if $x, Bx, B^2x, \ldots,$ are all simply normal in all of the scales $B, B^2, B^3, \ldots$. That Borel's definition is equivalent to Definition 1.2 was first proved by Niven and Zuckerman **(7)**, and later a very simple proof was given by Cassels **(4)**. Definition 1.3 is essentially that of Besicovitch **(1)**, differing only in trivial details which do not affect the validity of Besicovitch's results. Definition 1.4 is that of the writer.

In §2 we shall show, first, how the problem of the (k, ϵ)-normality of almost all of an increasing sequence of integers can be reduced to the case $k = 1$ (Theorem 2.1). Also the following problem is treated: Consider an increasing

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sequence $\{a_n\}$ of positive integers expressed in some fixed scale B, such that, for any given k and any given $\epsilon > 0$, almost all of the integers a_n are (k, ϵ)-normal in the scale B. Under what sufficient condition can we conclude that the number $.a_1a_2a_3\ldots$, formed by writing the integers a_n in order and in juxtaposition after the decimal point, is a normal number? (See Theorem 2.2.) Finally, in §3, we shall show how to construct quasi-normal numbers out of normal numbers (Theorem 3.1). For these quasi-normal numbers the asymptotic frequency of any k-digit sequence is actually determined (Theorem 3.2); and we obtain the result that quasi-normality does not imply normality (Theorem 3.3).

2. Some relations between ϵ-normality, (k, ϵ)-normality, and normality. Besicovitch **(1)** proved that, for any given integer k and any given $\epsilon > 0$, almost all integers are (k, ϵ)-normal, and almost all squares of integers are ϵ-normal. That the squares of almost all integers are also (k, ϵ)-normal is a particular case of a theorem proved by Davenport and Erdös **(5)**. We shall prove that in a quite general way the problem of (k, ϵ)-normality reduces to one of ϵ-normality.

LEMMA 2.1. *Given $\epsilon > 0$ and an integer $k > 2$. A sufficient condition that an integer*

$$m = a_{\mu-1}a_{\mu-2}\ldots a_1a_0 \qquad (a_{\mu-1} \neq 0), \tag{1}$$

be (k, ϵ)-normal in the scale B is that m be ϵ'-normal in the scale B^r, where $k/r < \epsilon/3$, $\epsilon' B^r < \epsilon/3$, and that m be sufficiently large that $r/\mu < \epsilon/3B^k$.

Proof. Let m be an integer which is ϵ'-normal in the scale B^r, where r, ϵ', and μ satisfy the hypothesis.

The digits of the scale B^r, when represented in the scale B, constitute the complete set of r-digit sequences of the scale B (if we write, where necessary, initial zeros). Let $b_1b_2\ldots b_k$ be any given k-digit sequence of the scale B. Then $b_1b_2\ldots b_k$ occurs in the complete set of r-digit sequences of the scale B exactly $(r-k+1)B^{r-k}$ times.

Let us represent m in the scale B^r,

$$m = A_{\nu-1}A_{\nu-2}\ldots A_1A_0 \qquad (A_{\nu-1} \neq 0), \tag{2}$$

where $\mu/r \leq \nu < \mu/r + 1$. Let us then replace each A_i by the corresponding sequence of r digits of the scale B. We obtain, thus, a representation of m in the form

$$m = 00\ldots 0a_{\mu-1}a_{\mu-2}\ldots a_1a_0, \tag{3}$$

where the number of initial zeros is less than r.

Since m is ϵ'-normal in the scale B^r, every digit of the scale B^r occurs in (2) more than $(B^{-r} - \epsilon')$ times. Hence $b_1b_2\ldots b_k$ occurs in (3) more than

$$(B^{-r} - \epsilon')\frac{\mu}{r}(r-k+1)\,B^{r-k}$$

times. In this estimate we disregard possible occurrences of $b_1b_2 \ldots b_k$ beginning in some A_i and ending in A_{i-1}. If we also take account of the fact that, in going from (3) to (1), less than r sequences of k digits can be lost, we see that

$$\begin{aligned} N(m, b_1b_2 \ldots b_k) &> \left(\frac{1}{B^r} - \epsilon'\right) \frac{\mu(r-k+1)}{r} B^{r-k} - r, \\ &> \left(\frac{1}{B^k} - \frac{\epsilon' B^r}{B^k} - \frac{k}{rB^k} - \frac{r}{\mu}\right)\mu \\ &> \left(\frac{1}{B^k} - \frac{\epsilon}{B^k}\right)(\mu - k + 1). \end{aligned}$$

This is true for every k-digit sequence. It follows that, for every k-digit sequence, $b_1b_2 \ldots b_k$,

$$N(m, b_1b_2 \ldots b_k) < (B^{-k} + \epsilon)(\mu - k + 1),$$

and hence m is (k, ϵ)-normal in the scale B.

THEOREM 2.1. *Let $\{a_n\}$ be an increasing sequence of positive integers having the property that, for any given $\epsilon > 0$ and any given scale B, almost all a_n are ϵ-normal. Then the sequence $\{a_n\}$ has the property that, for any given $\epsilon > 0$, any given $k \geqslant 2$, and any given scale B, almost all a_n are (k, ϵ)-normal.*

Proof. For a given $\epsilon > 0$, a given k, and a given scale B, choose an r and an $\epsilon' > 0$ which satisfy the hypothesis of Lemma 2.1. By the hypothesis of the theorem, almost all a_n are ϵ'-normal in the scale B^r. Choose also a μ which satisfies the hypothesis of the lemma. Almost all a_n have at least μ digits. It follows that almost all a_n satisfy the entire hypothesis of the lemma, and hence are (k, ϵ)-normal in the scale B.

THEOREM 2.2. *Let $\{a_n\}$ be an increasing sequence of positive integers having the property that, for any given k and any given $\epsilon > 0$, almost all a_n are (k, ϵ)-normal in the scale B. Let ν_i denote the number of digits in a_i $(i = 1, 2, 3, \ldots)$, and let*

$$S_n = \sum_{i=1}^{n} \nu_1.$$

Then a sufficient condition that the number $x = .a_1a_2a_3 \ldots$ be normal in the scale B is that

$$n\nu_n = O(S_n). \tag{4}$$

Proof. Let $b_1b_2 \ldots b_3$ be any given sequence of k digits of the scale B. Let m be an integer and let n be such that $S_n \leqslant m < S_{n+1}$. Let $N_m(x, b_1b_2 \ldots b_k)$ denote the number of occurrences of $b_1b_2 \ldots b_k$ in the first m digits of x. Then, for a given $\epsilon > 0$,

$$N_m(x, b_1b_2 \ldots b_k) \geqslant (B^{-k} - \epsilon) \sum_{\lambda}{}'(\nu_\lambda - k + 1),$$

where $\sum'$ is taken over the values of $\lambda \leqslant n$ for which a_λ is (k, ϵ)-normal.

Let the number of integers among $a_1, a_2, \ldots, a_n$ which are not (k, ϵ)-normal be denoted by ω_n. By hypothesis $\omega_n = o(n)$ as $n \to \infty$. Also $\nu_\lambda < \nu_n$ for every $\lambda < n$. Hence

$$N_m(x, b_1b_2 \ldots b_k) > (B^{-k} - \epsilon)\{\sum_\lambda{}' \nu_\lambda - (n - \omega_n)(k-1)\}$$
$$> (B^{-k} - \epsilon)(S_n - \omega_n\nu_n - nk),$$

and

$$\frac{N_m(x, b_1b_2 \ldots b_k)}{m - k + 1} > (B^{-k} - \epsilon)\frac{S_n - \omega_n\nu_n - nk}{S_{n+1}}$$
$$= (B^{-k} - \epsilon)\left\{1 - \frac{1}{n+1}\cdot\frac{(n+1)\nu_{n+1}}{S_{n+1}}\right\}\left(1 - \frac{\omega_n}{n}\cdot\frac{n\nu_n}{S_n} - \frac{k}{\nu_n}\cdot\frac{n\nu_n}{S_n}\right),$$

which, by (4), approaches $B^{-k} - \epsilon$ as $n \to \infty$. Hence

$$\liminf_{m\to\infty} \frac{N_m(x, b_1b_2 \ldots b_k)}{m - k + 1} > B^{-k} - \epsilon,$$

and, since ϵ is arbitrary,

$$\liminf_{m\to\infty} \frac{N_m(x, b_1b_2 \ldots b_k)}{m - k + 1} > B^{-k}.$$

Since this is true for every k-digit sequence, we have

$$\lim_{m\to\infty} \frac{N_m(x, b_1b_2 \ldots b_k)}{m - k + 1} = B^{-k},$$

and x is normal in the scale B.

Remark 1. It is easily seen that a sufficient condition for (4) is that $\nu_n = O(\log n)$. For, noting that

$$\nu_n = 1 + [\log_B a_n] > \frac{\log n}{\log B},$$

we see that

$$\frac{n\nu_n}{S_n} < \frac{(C \log B)\, n \log n}{\log 2 + \log 3 + \ldots + \log n} = \frac{(C \log B)\, n \log n}{n \log n + o(n \log n)},$$

which approaches $C \log B$ as $n \to \infty$.

The condition $\nu_n = O(\log n)$ is satisfied, for example, if $a_n = [f(n)]$, where $f(x)$ is a polynomial with real coefficients, $f(n) > 0$ for positive values of n.

It can easily be shown, also, that (4) holds if $\mu_1 n^\alpha < \nu_n < \mu_2 n^\alpha$ ($n = 1, 2, 3, \ldots$), where μ_1, μ_2, and α are positive constants.

Remark 2. If, however, the a_n increase too rapidly the conclusion of Theorem 2.2 does not hold. To show this we shall employ a sequence of integers, $\{I_m\}$, expressed in some scale B, which, for each m are constructed as follows: Write first m consecutive equal digits $B - 1$, and follow these at each successive position by the smallest digit of the scale B which does not cause the

repetition of a previously occurring sequence of m digits, continuing thus until no longer possible. It is not difficult to see that the integer thus constructed contains every m-digit sequence of the scale B exactly once and hence consists of exactly $B^m + m - 1$ digits. (For the case $B = 2$, see Lessard, Problem 4385, American Mathematical Monthly, 58 (1951), 573-574.) It can also easily be ascertained that, for any given k and any given $\epsilon > 0$, the integers I_m for almost all m, in fact, for all except finitely many m, are (k, ϵ)-normal in the scale B. (Sequences of digits which contain every m-digit sequence of the scale of representation exactly once have been investigated by Goode **(6)** and Rees **(8)**, who give methods of construction different from that above, and by de Bruijn **(3)**, who proves that, for the scale 2, the number of such sequences is $2^{f(m)}, f(m) = 2^{m-1}$, if cyclic permutations are accounted distinct.)

Let $J_m = I_m$ if m is not a perfect square, and let J_m be $B^m + m - 1$ consecutive equal digits $B - 1$ if m is a perfect square. Then the sequence $\{J_m\}$ has the property that, for any $k \geqslant 1$ and any $\epsilon > 0$, almost all J_m are (k, ϵ)-normal. But the number $.J_1J_2J_3\ldots$ is not even simply normal, for a quite simple estimate will show that, for the particular digit $(B - 1)$,

$$\limsup_{n\to\infty} \frac{N_n(x, B-1)}{n} \geqslant \frac{B-1}{B},$$

which is greater than $1/B$ if $B > 2$. For $B = 2$, it can be shown, by a closer estimate, that

$$\limsup_{n\to\infty} \frac{N_n(x, 1)}{n} \geqslant \frac{3}{4}.$$

3. Quasi-Normal Numbers. We shall show first that every number which is normal in the scale B is also quasi-normal in the scale B (see Definition 1.4). This follows from the following lemma.

LEMMA 3.1. *If x is normal in the scale B, and k, j, and i are any positive integers, and $b_1b_2\ldots b_k$ is any sequence of k digits of the scale B; then $b_1b_2\ldots b_k$ occurs in x in a position*[1] *congruent to i modulo j with the asymptotic frequency $1/jB^k$.*

Proof. Let r be the smallest integer for which $rj \geqslant k$. Then, by the well known property of normal numbers, every sequence of rj digits occurs in x in a position congruent to 1 modulo rj with the asymptotic frequency $1/rjB^{rj}$. Among the sequences of rj digits each, B^{rj-k} begin with $b_1b_2\ldots b_k$. Hence the asymptotic frequency of $b_1b_2\ldots b_k$ in x in a position congruent to 1 modulo rj is $1/rjB^k$. Applying the same principle to any of the numbers $B^{\rho-1}x$ ($\rho = 1, 2, 3, \ldots, rj$), we find that, for a fixed ρ $(1 \leqslant \rho \leqslant rj)$, the asymptotic frequency of $b_1b_2\ldots b_k$ in x in a position congruent to ρ modulo rj is also $1/rjB^k$. Since

[1]We shall say that a k-digit sequence occurs in a position congruent to i modulo j if the index of the first digit of the sequence is congruent to i modulo j.

there are r values of ρ that are congruent to i modulo j, it follows that the asymptotic frequency of $b_1b_2 \ldots b_k$ in x in a position congruent to i modulo j is $1/jB^k$.

The statement that every normal number is also quasi-normal is merely the particular case of Lemma 3.1 for $k = 1$.

In the proof of the next theorem we shall make use of two results obtained by Wall (9); first, the equivalence of the normality of a number x in the scale B and the uniform distribution modulo 1 of the sequence $\{B^n x\}$; and, second, the fact that if x is normal in the scale B, then x/s is normal in the scale B for every positive integer s.

It will be convenient to introduce the following notation: If X is any real number and q is a positive integer, then we mean by res $X(\text{mod } q)$ the number σ, where $0 \leqslant \sigma < q$ and $(X - \sigma)/q$ is an integer.

THEOREM 3.1. *Let x be a number which is normal in the scale B. Let s be any integer greater than* 1. *Let r_j =* res $[B^j x]$ (mod s). *Let n_j denote the number of digits preceding the jth occurrence of any given k-digit sequence $b_1b_2 \ldots b_k$ in x. Then*

(i) *the number $y = .r_1r_2r_3 \ldots$ is quasi-normal in the scale s;*

(ii) *the number $.r_{n_1}r_{n_2}r_{n_3} \ldots$ is simply normal in the scale s.*

Proof. By Wall's result (9), x/s is normal in the scale B, and, consequently, $\{B^n x/s\}$ is uniformly distributed modulo 1.

The number res $B^n x/s$ (mod 1), which is the fractional part of $B^n x/s$, for each value of n falls into one of the intervals $(0, 1/s)$, $(1/s, 2/s)$, $\ldots$, $(1\text{-}1/s, 1)$, namely, into the interval $(\sigma/s, (\sigma + 1)/s)$, where σ = res $[B^n x]$ (mod s). Since $\{B^n x/s\}$ is uniformly distributed modulo 1, the number of numbers, res $B^n x/s$ (mod 1), in each of the above intervals is asymptotically equal to n/s, and, consequently, the integers, res $[B^n x]$ (mod s) are asymptotically equally distributed among the integers $0, 1, 2, \ldots, s - 1$. Hence $y = .r_1r_2r_3 \ldots$ is simply normal in the scale s.

Further, $B^{u-t}x$ is normal in the scale B^t, where u and t are any positive integers. Let us take $u \leqslant t$. Then if $x = .a_1a_2a_3 \ldots$ in the scale B, $B^{u-t}x = .00 \ldots 0a_1a_2a_3 \ldots$ in the scale B, where the number of initial zeros is $t - u$. In the scale B^t, $B^{u-t}x = .A_1A_2A_3 \ldots$, where A_1, represented in the scale B is $00 \ldots 0a_1a_2 \ldots a_u$, and A_j, $j > 1$, is $a_{u+(j-2)t+1} \ldots a_{u+(j-1)t}$.

If we write R_j = res $[B^{jt}(B^{u-t}x)]$ (mod s), then, by the preceding argument, $.R_1R_2R_3 \ldots$ is simply normal in the scale s. But $R_j = r_{u+(j-1)t}$. Hence the number $.r_ur_{u+t}r_{u+2t} \ldots$ is simply normal in the scale s and y is quasi-normal in the scale s.

For the proof of (ii), consider those values of n for which the numbers res $B^n x$ (mod 1) lie in the interval of length $1/B^k$ whose left endpoint is $.b_1b_2 \ldots b_k$. Note that these are precisely the values $n_1, n_2 \ldots$, defined in the statement of the theorem. For each of these values of n, the number res $B^n x/s$

(mod 1) lies in one of s non-overlapping intervals of length $1/sB^k$. Of these intervals, one lies in each of the intervals $(\sigma/s, (\sigma + 1)/s)$; and, for each n, the value of σ is determined by

$$\sigma = \operatorname{res}[B^n x](\operatorname{mod} s).$$

Since there are asymptotically n/sB^k numbers res $B^n x/s$ (mod 1) in each of these intervals of length $1/sB^k$, it follows that the values of res $[B^n x]$ (mod s) for the values $n_1, n_2, \ldots$, are asymptotically equally distributed among the integers $0, 1, 2, \ldots, s - 1$, and, therefore, the number $.r_{n_1} r_{n_2} r_{n_3} \ldots$ is simply normal in the scale s.

Remark. Note that it follows from Theorem 3.1 that the asymptotic frequency of a given digit of the scale s in y in the positions n_j is $1/sB^k$.

The numbers y, as constructed in Theorem 3.1, not only are quasi-normal, but they possess the additional property (ii), and this additional property enables us to calculate, for this class of quasi-normal numbers, the asymptotic frequency of any k-digit sequence.

THEOREM 3.2. *Let x be normal in the scale B; let s be any integer greater than* 1; *and let $\gamma_1\gamma_2 \ldots \gamma_k$ be any given k-digit sequence in the scale s. Then the asymptotic frequency of $\gamma_1\gamma_2 \ldots \gamma_k$ in the number y, as defined in Theorem* 3.1, *is equal to*

$$\frac{1}{sB^{k-1}} \prod_{i=2}^{k} \left(\left[\frac{B}{s} \right] + \delta_i \right).$$

Here $\delta_i = 0$ or 1, according as $\mu_i > m$ or $\mu_i < m$, where $m = $ res B (mod s) *and $\mu_i = $* res $(\gamma_i - \gamma_{i-1}B)$(mod s).

Proof. Let $c_1c_2 \ldots c_k$ be any k-digit sequence of the scale B, and let r denote the integer res $[B^n x]$ (mod s), where n is the number of digits in x preceding an occurrence of $c_1c_2 \ldots c_k$.

We inquire, how many combinations consisting of a value of r and a sequence $c_1c_2 \ldots c_k$ in x will result in the occurrence of the given sequence $\gamma_1\gamma_2 \ldots \gamma_k$ in the corresponding position in y?

For each of the digits of the sequence $c_1c_2 \ldots c_k$ and each of the values of r, the following relations must hold:

$$rB + c_1 \equiv \gamma_1 (\operatorname{mod} s), \tag{6}$$

$$\begin{cases} \gamma_1 B + c_2 \equiv \gamma_2 (\operatorname{mod} s), \\ \gamma_2 B + c_3 \equiv \gamma_3 (\operatorname{mod} s), \\ \quad - \quad - \quad - \\ \gamma_{k-1} B + c_k \equiv \gamma_k (\operatorname{mod} s), \end{cases} \tag{7}$$

where $0 \leq r < s$, $0 \leq c_i < B$.

The left member of (6) takes on the sB values $0, 1, 2, \ldots, sB - 1$, as r and c_1 range independently over the integers from 0 to $s - 1$ and from 0 to $B - 1$, respectively, of which values exactly B are congruent to γ_1 modulo s.

The relations (7) are independent of each other and of (6). For each of the relations (7), it is easily seen that there are either $[B/s] + 1$ or $[B/s]$ values of c_i which satisfy the relation, according as $m > \mu_i$ or $m \leq \mu_i$, where m and μ_i are defined as in the statement of the theorem. Hence the number of combinations of r and sequences $c_1c_2 \ldots c_k$ which result in the occurrence of a given sequence $\gamma_1\gamma_2 \ldots \gamma_k$ in y is

$$B\prod_{i=2}^{k}\left(\left[\frac{B}{s}\right]+\delta_i\right),$$

where δ_i is defined as in the statement of the theorem.

From the remark following Theorem 3.1, if follows, then, that the asymptotic frequency of $\gamma_1\gamma_2 \ldots \gamma_k$ in y is equal to

$$\frac{1}{sB^{k-1}}\prod_{i=2}^{k}\left(\left[\frac{B}{s}\right]+\delta_i\right).$$

THEOREM 3.3. *The number y, defined as in Theorem* 3.1, *is normal in the scale s if and only if s divides B.*

Proof. If s divides B, then $m = 0$, and each factor of the product in Theorem 3.2 is B/s. Thus the asymptotic frequency of any k-digit sequence in y is $1/s^k$, and y is normal in the scale s.

If s does not divide B, then in order that the asymptotic frequency of a given k-digit sequence in y be $1/s^k$, we must have

$$\prod_{i=2}^{k}\left(\left[\frac{B}{s}\right]+\delta_i\right)=\left(\frac{B}{s}\right)^{k-1}.$$

But the left member of this equation is an integer, while the right member is not unless $k = 1$.

Thus we have answered in the negative the question whether a quasi-normal number is necessarily normal. Indeed, with regard to the class of quasi-normal numbers y derived in the manner of Theorem 3.1 from a normal number x, we can say that if s does not divide B, then for no $k > 1$ does any k-digit sequence of y have the proper asymptotic frequency. We note, too, that if $s > B$, then by (7) there are some sequences of digits of the scale s which do not occur in y at all, in particular, any sequence in which a zero is followed by a digit equal to or greater than B.

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ON THE MODULAR REPRESENTATIONS OF THE SYMMETRIC GROUP PART IV

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1. Introduction. The study of the modular representation theory of the symmetric group has been greatly facilitated lately by the introduction of the *graph* (**9**, III), the *q-graph*[1] (**5**) and the *hook-graph* (**4**) of a Young diagram $[\lambda]$. In the present paper we seek to coordinate these ideas and relate them to the *r-inducing* and *restricting* processes (**9**, II).

If we denote the number of nodes of class r which can be added to or removed from $[\lambda]$ by d and d^* respectively, then the Main Theorem 6.3 expresses the change in weight of $[\lambda]$, which arises as a result of r-inducing or restricting, in terms of d and d^*. Further explicit results connect d and d^* with the corresponding δ, δ^* associated with the q-core of $[\lambda]$, which are illustrated in Tables I and II at the end of the paper.

It is interesting to note that the set of Young diagrams thus associated with a given $[\lambda]$ constitutes a Boolean Algebra of *dimension* $d + d^*$, whose partial ordering is that established by r-inducing. Two diagrams, or elements of the Boolean Algebra, of the same dimension d^* have the same weight w. Moreover, *dual* elements also have the same weight, and this shows itself in the symmetry of Tables I and II.

That these results are so explicit is somewhat surprising. No attempt is made here to apply them to the study of the structure of the indecomposables of the regular representation of S_n, this being left to a subsequent paper.

2. The graph $G[\lambda]$ and the q-graph $\mathbf{G}[\lambda]$. We begin by introducing the notion of the graph of a Young diagram $[\lambda] = [\lambda_1, \lambda_2, \ldots \lambda_m]$ obtained by replacing the (i, j) node of $[\lambda]$ by

2.1 $$g_{i,j} = j - i.$$

We shall denote this *graph* $(g_{i,j})$ by $G[\lambda]$. The quantity $1/\rho$ appearing in Young's semi-normal representation of S_n is given (**9**, III) by

2.2 $$\frac{1}{\rho} = g_{i,j} - g_{k,l},$$

where $i < k$ and $j > l$

If we reduce $g_{i,j}$ modulo q and require that the residue be non-negative, i.e. set

2.3 $$g_{i,j} \equiv \mathbf{g}_{i,j} \pmod{q}, \qquad 0 \leq \mathbf{g}_{i,j} < q,$$

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[1]As in (**7**, **11**) we use q instead of p to indicate that q may be composite.

we obtain D. E. Littlewood's *q-graph* ($\mathfrak{g}_{i,j}$) which we shall denote by $\mathbf{G}[\lambda]$. An immediate consequence of 2.1 and 2.3 is the relation

2.4 $$\mathfrak{g}_{i,j-1} = \mathfrak{g}_{i+1,j} \equiv \mathfrak{g}_{i,j} - 1 \pmod q,$$

from which it follows that

2.5 *Any right or skew hook of* $\mathbf{G}[\lambda]$ *of length kq with head node of class r, is made up of a succession of residues*

$$r, r-1, \ldots, 1, 0, q-1, \ldots, 1, 0, q-1, \ldots, r+2, r+1,$$

each residue appearing k times.

Thus we may associate the class of its head node **(10)** with any kq-hook of $[\lambda]$. The significance of this association so far as the *star diagram* **(3, 4)** or *q-quotient* $[\lambda]_q$ of $[\lambda]$ is concerned will appear shortly. The leg length of such a hook will depend on the core.

It follows from 2.5 that the residue content of $\mathbf{G}[\lambda]$ is uniquely determined by that of the core and the *weight* w of $[\lambda]$, which is the number of removable q-hooks. Littlewood proved the following important result **(5**, p. 337**)**:

2.6 *A necessary and sufficient condition that two diagrams* $[\lambda']$ *and* $[\lambda'']$ *have the same weight and the same q-core is that* $\mathbf{G}[\lambda']$ *and* $\mathbf{G}[\lambda'']$ *contain the same set of residues modulo q.*

Another approach to the problem is to consider the hooks with corner nodes in the first column of $[\lambda]$, setting

2.7 $$l_i = \lambda_i + m - i,$$

where m is the number of rows of $[\lambda]$. The following theorem[2] supplements 2.6 and makes it possible to actually construct the core of a diagram, given the residue content of its q-graph.

2.8 *A diagram is a q-core if and only if each class of congruent* l_i*'s contains all smaller non-negative integers congruent to the largest one in the class, the* 0*-class being empty.*

The details of this construction are being given elsewhere.

3. r-inducing and r-restricting. The reciprocity theorem of Frobenius is of deep significance in the representation theory of finite groups over a field of characteristic zero. The relation between *inducing* and *restricting* thus provided is particularly simple in the case of the symmetric group S_{n+1} if the subgroup under consideration is taken to be S_n.

Consider first the *inducing* process, taking the irreducible representation $[\lambda]$ of S_n, to yield the reducible representation **(7; 8)**

$$[\lambda] . [1]$$

[2]As stated in **(11)**, the theorem was not quite correct.

of S_{n+1} whose irreducible components are obtained by adding a node to $[\lambda]$ in all possible ways. For example:

3.1 $$[3, 2, 1] \uparrow [4, 2, 1] + [3^2, 1] + [3, 2^2] + [3, 2, 1^2].$$

Conversely, if we take an irreducible representation $[\lambda]$ of S_n and *restrict* it to the operations of S_{n-1}, the irreducible components of the resulting representation of S_{n-1} will be obtained by removing a node from $[\lambda]$ in all possible ways. For example:

3.2 $$[4, 2, 1] \downarrow [3, 2, 1] + [4, 1^2] + [4, 2].$$

The two symbols $\uparrow \downarrow$ are convenient to indicate inducing and restricting, respectively, particularly in the modular case to which we now proceed.

If we think of the processes as operating on $\mathbf{G}[\lambda]$ instead of on $[\lambda]$, we may distinguish the residue class of the added node by inserting an r above or below the arrow. Thus we may add a node of class r *only* and designate the process as *r-inducing*. For example, taking $q = 3$, $r = 0$,

3.3 $$[3, 2, 1] \overset{0}{\uparrow} [4, 2, 1] + [3, 2, 1^2].$$

Similarly, we may limit the restricting process to *r-restricting*, so that

3.4 $$[4, 2, 1] \underset{0}{\downarrow} [3, 2, 1] + [4, 1^2].$$

What is the significance of these limited processes as regards the modular representation theory of S_n? We state the following modification of 2.6:

3.5 *The necessary and sufficient condition that two diagrams* $[\lambda']$ *and* $[\lambda'']$ *obtained by adding (removing) a node to(from) a given diagram* $[\lambda]$ *should have the same q-core is that the added(removed) nodes should be of the same residue class in* $\mathbf{G}[\lambda]$.

While 3.5 is not essential in the application of the inducing or restricting processes, since one may readily determine the q-core of a diagram **(1, 2, 6)**, nevertheless the simplification thus introduced makes it possible to keep track of changes in the star diagram or the q-quotient $[\lambda]_q$ and consequently in the weight of $[\lambda]$. We shall study these changes in detail with the aid of the hook-graph described in the following section. We prove here an important preliminary result, after making the following

DEFINITIONS. We shall call[3]

(i) the number d of nodes of class r which can be added to $[\lambda]$ the *r-defect* of $[\lambda]$;

(ii) the number d^* of nodes of class r which can be removed from $[\lambda]$ the *r-affect* of $[\lambda]$;

denoting by δ, δ^* the *r-defect* and *r-affect* of the q-core of $[\lambda]$.

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[3]The r-defect must not be confused with Brauer's *defect* group or *defect* of a block **(1)**.

3.6 *Neither adding nor removing a kq-hook of class different from r or $r - 1$ changes d, d^*, δ, δ^*.*

Proof. Since the class of a kq-hook is defined to be the class of its head node, a hook of class different from r or $r - 1$ cannot begin or end in a node of class r. Thus a node of class r must be *internal* to such a skew hook, and if it could have been removed from $[\lambda]$, the addition or removal of such a hook does not affect this possibility. Similarly, if a node of class r could be added to $[\lambda]$, the addition or removal of a kq-hook of class different from r or $r - 1$ does not affect the possibility of such an addition. Moreover, the core remains the same so d, d^*, δ, δ^* remain unchanged.

For convenience, we shall abbreviate "a node of class r" to an *r-node.* Similarly we shall describe the position such a node may occupy in $\mathbf{G}[\lambda]$ as an *r-position.*

4. The hook-graph $H[\lambda]$. Since the hook structure of $[\lambda]$ is different for every q, it is not only convenient but also of general significance, to make all such computations once and for all **(4)**. To this end we set in place of the (i, j) node of $[\lambda]$ the quantity

4.1 $$h_{i,j} = (\lambda_i - j) + (\lambda'_j - i) + 1,$$

where $[\lambda']$ is the transpose of $[\lambda]$. Clearly, $h_{i,j}$ is the length of the right hook having its corner at the (i,j) node of $[\lambda]$. We denote the *hook-graph* $(h_{i,j})$ by $H[\lambda]$. Note that $l_i = h_{i,1}$ if $m = \lambda'_1$ in 2.7.

We have immediately from 2.1 and 4.1 that

4.2 $$\begin{aligned} h_{i,k} - h_{j,k} &= (\lambda_i - i) - (\lambda_j - j) \\ &= g_{i,j} + \lambda_i - \lambda_j, \end{aligned}$$

so that this difference is independent of k, which provides a useful check on the construction of $H[\lambda]$. The following relation between $\mathbf{G}[\lambda]$ and $H[\lambda]$ is fundamental in all that follows:

4.3 *If in* $\mathbf{G}[\lambda]$ *the ith row ends in s and the jth column in t, then in* $H[\lambda]$

$$h_{i,j} \equiv s - t + 1 \qquad (\text{mod } q).$$

Proof. From 4.1 we have

$$\begin{aligned} h_{i,j} &= (\lambda_i - j) + (\lambda'_j - i) + 1 \\ &= (\lambda_i - i) - (j - \lambda'_j) + 1 \\ &= g_{i,\lambda_i} - g_{\lambda'_j,j} + 1, \end{aligned}$$

so that by 2.3 we have the desired result:

$$h_{i,j} \equiv \mathbf{g}_{i,\lambda_i} - \mathbf{g}_{\lambda'_j,j} + 1 \qquad (\text{mod } q).$$

Clearly an r-node can be added at the end of a row of $\mathbf{G}[\lambda]$ whose final node is of class $r - 1$, provided such an r-position is also at the foot of a column whose final node is of class $r + 1$, and only in such places. But the $h \equiv 0 (\text{mod } q)$ which yield the constituent of $[\lambda]_q$ of class $r - 1$ lie in rows which end in $(r - 1)$

-nodes and consequently in columns which end in r-nodes, and those which yield the constituent of class r lie in rows which end in r-nodes and in columns which end in $(r+1)$-nodes. Thus the addition to $\mathbf{G}[\lambda]$ of an r-node modifies one or both of these constituents of $[\lambda]_q$. On the other hand, the addition of an r-node cannot affect the other constituents of $[\lambda]_q$. A similar argument applies to the removal of an r-node, proving the following analogue of 3.6:

4.4 *Neither adding nor removing an r-node changes the constituents of $[\lambda]_q$ of class different from r or $r-1$, but does modify one or both of these constituents.*

In the following sections we shall study the effects of r-inducing and restricting so far as the *weight* is concerned. To simplify matters we might assume that $[\lambda]_q$ has only constituents of class r and $r-1$, in view of 3.6 and 4.4. However, for the considerations of this paper such an assumption is unnecessary.

5. The change in weight ∇. Consider any diagram $[\lambda]$ with r-defect $d > 0$ so that we may add an r-node at some r-position P at the intersection of the ith row and jth column of $\mathbf{G}[\lambda]$. The effect on $[\lambda]_q$ will be two-fold.

(i) Consider first those $h \equiv -1 (\text{mod } q)$ in $H[\lambda]$ which are thereby changed into $h \equiv 0 (\text{mod } q)$. Setting $s = r - 1$ in 4.3 it follows that the number of $h_{i,k} \equiv -1 (\text{mod } q)$ for $k < j$ is equal to the number of foot-nodes in $\mathbf{G}[\lambda]$ of class $r+1$ *below* P; denote this number by $(r+1)_{fB}$. On the other hand, the number of $h_{l,j} \equiv -1 (\text{mod } q)$ for $l < i$, which lie in the jth column, is similarly obtained by setting $t = r + 1$ in 4.3, and is the number of head-nodes of class $r - 1$ lying *above* P, which we may denote by $(r-1)_{hA}$. Thus adding an r-node at P leads to an *increase* in the weight of $[\lambda]$ by an amount

$$5.1 \qquad \Delta = (r+1)_{fB} + (r-1)_{hA}.$$

(ii) The second effect of adding an r-node at P is to change those $h \equiv 0 (\text{mod } q)$ which appear in the ith row and the jth column of $H[\lambda]$ into $h \equiv 1 (\text{mod } q)$. As before, it follows that the number of $h_{i,k} \equiv 0 (\text{mod } q)$ for $k < j$ is equal to the number of foot-nodes of class r *below* P, which number we denote by $(r)_{fB}$. On the other hand, the number of $h_{l,j} \equiv 0 (\text{mod } q)$ for $l < i$, which lie in the jth column is equal to the number of head-nodes of class r which lie *above* P, which number we denote by $(r)_{hA}$. Thus adding, an r-node at P leads to a *decrease* in the weight of $[\lambda]$ by an amount

$$5.2 \qquad \bar{\Delta} = (r)_{fB} + (r)_{hA}.$$

Combining the two effects we see that the total change in the weight of $[\lambda]$ caused by adding an r-node in the r-position P is given by

$$5.3 \qquad \nabla = \Delta - \bar{\Delta} = \{(r+1)_{fB} + (r-1)_{hA}\} - \{(r)_{fB} + (r)_{hA}\}.$$

If we compare the result of adding an r-node in two different r-positions P' and P'' to yield two different q-graphs $\mathbf{G}[\lambda']$ and $\mathbf{G}[\lambda'']$, then we know by 3.5

that $[\lambda']$ and $[\lambda'']$ have the same q-core and the same weight, and ∇ has the same value in each case. Effectively, 3.5 states that adding an r-node to $\mathbf{G}[\lambda]$ can be passed back through the removable hooks to the core, and that any change in weight is due to the effect of such an addition on the core. It is not without interest to follow through the changes in the terms on the right-hand side of 5.3 which arise when the r-node is added at different r-positions P, or when a kq-hook is removed from $\mathbf{G}[\lambda]$ which does not begin or end at P, but we leave this to the reader.

We now examine briefly the effect of *removing* an r-node, and there is no loss of generality if we consider it to be the one previously added at P on the rim of $\mathbf{G}[\lambda]$ to yield $\mathbf{G}[\lambda']$; of course we shall obtain $\mathbf{G}[\lambda]$ again. As before, there are two effects to consider.

(iii) Those $h \equiv 0 \pmod q$ in $H[\lambda]$ which were changed into $h \equiv 1 \pmod q$ in $H[\lambda']$ by adding an r-node at P in (ii) are precisely those which now yield $h \equiv 0 \pmod q$ in the reverse process.

(iv) Similarly, those $h \equiv 0 \pmod q$ in $H[\lambda']$ are now changed into $h \equiv -1 \pmod q$ in $H[\lambda]$.

Thus r-restricting *interchanges* the roles of Δ and $\bar{\Delta}$ and so changes the sign of the difference ∇.

5.4 *The change in weight arising from the addition of an r-node to* $[\lambda]$ *is given by*

$$\nabla = \Delta - \bar{\Delta}$$

where

$$\Delta = (r+1)_{fB} + (r-1)_{hA}, \quad \bar{\Delta} = (r)_{fB} + (r)_{hA},$$

and ∇ may be positive or negative. Similarly, the change in weight arising from removing an r-node from $[\lambda]$ *is given by* $-\nabla$.

6. An explicit formula for ∇. While the results of the preceding section are complete they do not express ∇ explicitly in terms of $[\lambda]$. To do this we study the functions Δ and $\bar{\Delta}$ in greater detail.

Consider first the function

5.2 $$\bar{\Delta} = (r)_{fB} + (r)_{hA}.$$

Certainly all removable r-nodes of $\mathbf{G}[\lambda]$ contribute to $\bar{\Delta}$, since each one is a possible head-node and(or) foot-node of a kq-hook. But other r-nodes contribute as well.

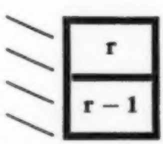

FIG. 1

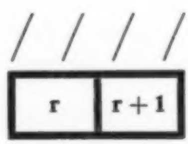

FIG. 2

We have illustrated in Figs. 1 and 2 parts of the rim of $\mathbf{G}[\lambda]$ in which no r-node is removable, and yet the arrangement in Fig. 1, appearing say ϵ_A times *above* P contributes ϵ_A to $(r)_{hA}$. Similarly, the arrangement in Fig. 2 appearing ϵ_B times *below* P contributes ϵ_B to $(r)_{fB}$. Thus

6.1 $$\bar{\Delta} = (r)_{hA} + (r)_{fB} = d^* + \epsilon_A + \epsilon_B.$$

On the other hand, no r-node can be added to the right of $r - 1$ in Fig. 1 and below $r + 1$ in Fig. 2. But the quantity

5.1 $$\Delta = (r + 1)_{fB} + (r - 1)_{hA}$$

enumerates not only $(r - 1)$-nodes in configurations such as Fig. 1 appearing ϵ_A times above P and $(r + 1)$-nodes in configurations such as Fig. 2 appearing ϵ_B times below P, but also all places where an r-node *can* be added, excluding the position P itself, so that

6.2 $$\Delta = (r + 1)_{fB} + (r - 1)_{hA} = (d - 1) + \epsilon_A + \epsilon_B.$$

It is to be noted that the epsilons depend on the choice of P on the rim of $[\lambda]$. Subtracting 6.1 from 6.2, these variable terms disappear and we have the desired explicit expression for ∇.

In the restricting process we must interchange the roles of Δ and $\bar{\Delta}$. An exactly analogous argument leads to the equations

$$\bar{\Delta}' = d^{*\prime} - 1 \ + \epsilon_A' + \epsilon_B',$$
$$\Delta' = d' \ + \epsilon_A' + \epsilon_B',$$

which, when subtracted, yield the change in weight $\nabla' = \bar{\Delta}' - \Delta'$. If we are considering the same r-position, first inducing and then restricting as at the end of §5, then

$$\begin{aligned}\nabla' &= \bar{\Delta}' - \Delta' = d^{*\prime} - d' - 1\\ &= (d^* + 1) - (d - 1) - 1\\ &= -(d - d^* - 1)\\ &= -(\Delta - \bar{\Delta}) = -\nabla,\end{aligned}$$

as in 5.4. We collect together these results in our

6.3 MAIN THEOREM. *The change in weight of* $[\lambda]$ *arising by adding an r-node is given by*

$$\text{(a)} \quad d - d^* - 1,$$

and by removing an r-node is given by

$$\text{(b)} \quad d^* - d - 1,$$

where d and d^* are, respectively, the r-defect and r-affect of $[\lambda]$.

The assumption that $[\lambda]$ is a p-core rules out the appearance of configurations such as Fig. 1 above any r-position and such as Fig. 2 below any r-position,

since otherwise a kq-hook beginning or ending to the left of, or above, P would be removable and $[\lambda]$ would not be a core. For a similar reason $\delta^* = 0$ if $\delta \neq 0$. So that 6.3 (a) becomes in this case

6.4 $$\nabla = \delta - 1,$$

for the addition of an r-node. If we restrict a core for which $\delta = 0$ with $\delta^* \neq 0$, a corresponding change takes place in 6.3 (b). We prove the following interesting result:

6.5 *If the r-defect of a q-core $[\lambda]$ is δ, then the addition of δ r-nodes to $[\lambda]$ yields a q-core $[\lambda']$.*

Proof. We need only consider the $h \equiv -1 (\mathrm{mod}\ q)$ which appear at the intersections of rows and columns ending in r-positions, the number of these positions being δ. Adding an r-node at each position changes each such $h \equiv -1 (\mathrm{mod}\ q)$ of $H[\lambda]$ into an $h \equiv +1 (\mathrm{mod}\ q)$ of $H[\lambda']$. No new $h \equiv 0 (\mathrm{mod}\ q)$ appear, by 4.4. Thus $[\lambda']$ must be a q-core as required.

We state the corresponding theorem for r-restricting without proof.

6.6 *If the r-affect of a q-core $[\lambda]$ is δ^*, then the removal of δ^* r-nodes from $[\lambda]$ yields a q-core $[\lambda']$.*

Taking 6.5 and 6.6 together we have:

6.7 *Every q-core is obtainable by adding to the zero core first δ nodes of class r, then δ' nodes of class r', and so on, two successive values of r being necessarily distinct.*

It should be noted that the sequence of such additions for different r is not uniquely determined, so that 6.7 does not lead to a generating function for cores. Consider for example the 3-core $[4, 2^2, 1^2]$. The sequence of additions of δ nodes may be any one of the following:

6.8 $$I_0\, I_1\, I_2^2\, I_1\, I_0^3\, I_2^2, \qquad I_0\, I_2\, I_1^2\, I_2\, I_0^3\, I_2^2, \qquad I_0\, I_2\, I_1^2\, I_0^2\, I_2^3\, I_0,$$

where I_r^n indicates the addition of n nodes of class r.

There is a restriction on the choice of r for r-inducing on a q-core:

6.9 *The r-defect δ (r-affect δ^*) of a given core $[\lambda]$ must vanish for at least one value of r.*

Proof. If the class of the end node in the first row of $\mathbf{G}[\lambda]$ is r, then no node of class r can be added to $\mathbf{G}[\lambda]$, since this would imply that a kq-hook could be removed from $[\lambda]$ beginning in the first row and ending above the supposed r-position. Thus $\delta = 0$ for at least one value of r. By a similar argument $\delta^* = 0$ for at least one value of r; this is also implied by 2.8.

7. The r-Boolean Algebra. The totality of diagrams obtained from a given diagram $[\lambda]$ by r-inducing and r-restricting at every stage constitutes a *Boolean*

Algebra which we shall denote by rBA. To see this it is convenient to introduce the r-affect d^* as a label, writing

$$[\lambda] = [\lambda^{d^*}],$$

and setting

7.1 $$d + d^* = l.$$

The diagram $[\lambda^0]$ for which $d^* = 0$ is the 0-element of rBA and $[\lambda^l]$ for which $d^* = l$ is the I-element of rBA. The *dimension* of rBA is l, while that of any given diagram is d^*. The operations $\cup$ and $\cap$ are defined in a natural manner. Clearly

$$[\lambda'] \cup [\lambda'']$$

is that diagram $[\lambda]$ of smallest dimension such that $\mathbf{G}[\lambda]$ contains $\mathbf{G}[\lambda']$ and $\mathbf{G}[\lambda'']$. Similarly

$$[\lambda'] \cap [\lambda'']$$

is that diagram $[\lambda]$ of largest dimension such that $\mathbf{G}[\lambda]$ is contained in both $\mathbf{G}[\lambda']$ and $\mathbf{G}[\lambda'']$. The existence and uniqueness of the diagram $[\lambda]$ follows in each case from the nature of our construction.

Since we are concerned here with the *weight* w and not with the linkage properties **(2)** of the diagrams of rBA, it is unnecessary to distinguish diagrams having the same dimension d^*, since all these have the same weight w, d, d^*, δ, δ^*. Tables I and II give the values of these parameters in two typical cases.

If we denote the r-defect and affect of $[\lambda^i]$ by d_i and d_i^* respectively, then the weight w_i of $[\lambda^i]$ can be obtained by repeated application of 6.3 and is readily seen to be given by one or other of the following expressions:

7.2 $$w_i = \sum_{j=0}^{i-1} (d_j - d_j^* - 1) = \sum_{j=l}^{i+1} (d_j^* - d_j - 1),$$

according as we induce from $[\lambda^0]$ upwards or restrict from $[\lambda^l]$ downwards.

From our definitions of d and d^* it follows immediately that

7.3 $$d_i - d_i^* = (d_{i+1} - d_{i+1}^*) + 2,$$

so that second differences of w are constant. Thus:

7.4 *If from the* 2^l *diagrams belonging to an r-Boolean Algebra a typical one be chosen of each dimension* d^*, *then these diagrams can be located on a line*

$$d + d^* = l,$$

when d is plotted against d^*, *or on a parabola*

$$w = w_l + d^*(l - d^*),$$

when w is plotted against d^*.

It should be noted that successive r-inducing(restricting) applied in *all* possible ways yields diagrams of dimension d^*, each with a multiplicity

$d^*!(d!)$. Counting each distinct diagram once only, the number of diagrams of dimension d^* is

$$\binom{l}{d^*}$$

so that the total number of elements of rBA is 2^l, as stated in the theorem.

That the addition or removal of an r-node *commutes* with the addition or removal of any kq-hook which does not begin or end at P leads to the relation

7.5 $$d - d^* = \delta - \delta^*.$$

Proof. Since the change in weight of $[\lambda]$ for r-inducing is given by $d - d^* - 1$, this change must be accounted for by a corresponding change in weight of the core of $[\lambda]$, which, by the same argument, amounts to $\delta - \delta^* - 1$. Thus the quantities in 7.5 must be equal.

We have noted the special properties of δ, δ^* in §6, namely that if $\delta \neq 0$, then $\delta^* = 0$ and conversely. From 7.5 we have[4]

7.6 $$\begin{aligned} \delta &= \tfrac{1}{2}\{d - d^* + |d - d^*|\}, \\ \delta^* &= \tfrac{1}{2}\{d^* - d + |d - d^*|\}. \end{aligned}$$

With each diagram $[\lambda^i]$ of dimension i is associated a unique *complement* $[\lambda^{l-i}]$ of dimension $l - i$, *dual* to it in rBA. The following relations express the fundamental property of this duality relation and explain the symmetry of the tables.

7.7 $$d_i^* = d_{l-i}, \quad \delta_i^* = \delta_{l-i}.$$

Proof. The first relation is immediate. Using this and 7.6 we have:

$$\begin{aligned} \delta_i^* &= \tfrac{1}{2}\{d_i^* - d_i + |d_i - d_i^*|\} \\ &= \tfrac{1}{2}\{d_{l-i} - d_{l-i}^* + |d_{l-i} - d_{l-i}^*|\} \\ &= \delta_{l-i}. \end{aligned}$$

The examples used to illustrate these ideas in Tables I and II, have been chosen to bring out two things. In the first place, the oddness or evenness of l determines whether there is or is not a level of r-inducing in rBA where the weight remains constant. In the second place:

7.8 $$d = \delta, \qquad d^* = \delta^*,$$

for the 0 and l-elements of rBA, and one of these equalities implies the other by 7.5 or 7.7. In Table I these elements are cores. When this is not the case, as in Table II, these elements have special properties which we shall not consider here. Thus

[4]Drawn to my attention by J. S. Frame.

7.9 *If $d = \delta \neq 0$ for a diagram $[\lambda]$ then $d^* = \delta^* = 0$ and $[\lambda]$ is the 0-element of an r-Boolean Algebra. Conversely, if $d^* = \delta^* \neq 0$ then $d = \delta = 0$ and $[\lambda]$ is the I-element of an r-Boolean Algebra. These conditions are necessary as well as sufficient.*

TABLE I

w	0	4	6	6	4	0
d^*	0	1	2	3	4	5
d	5	4	3	2	1	0
δ^*	0	0	0	1	3	5
δ	5	3	1	0	0	0

In Table I $[\lambda^0] = [8, 6, 4, 2]$, $[\lambda^5] = [9, 7, 5, 3, 1]$, $q = 3$, $r = 2$.

TABLE II

w	7	10	11	10	7
d^*	0	1	2	3	4
d	4	3	2	1	0
δ^*	0	0	0	2	4
δ	4	2	0	0	0

In Table II $[\lambda^0] = [5, 4^2, 3]$, $[\lambda^4] = [6, 5, 4^2, 1]$, $q = 2$, $r = 1$.

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A GENERALIZATION OF THE YOUNG DIAGRAM

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1. Introduction. The method of A. Young for finding the set of primitive idempotents of the group algebra of the symmetric group is classical; it was first given by Frobenius **(4)** using results of Young **(10** and **11)**. A concise account can be found in **(9)** and a very detailed treatment in **(6)**.

From the purely algebraic point of view Young's method consists of finding pairs of subgroups R and C of the symmetric group S_n so that if

$$P = \sum_{r \epsilon R} r, \quad N = \sum_{c \epsilon C} c\, \sigma(c),$$

where $\sigma(c) = \pm 1$ according as c is an even or odd permutation, then PN is a multiple of a primitive idempotent of the group algebra of S_n. This will be the case if R and C satisfy a condition of von Neumann. Below, in Lemma 1, we show that a more general formulation of his condition applicable to any group is possible in algebraic terms. An application of this new condition to the group $\mathrm{GL}(2, q)$ is given in §§5–8 of this paper. In Lemma 2 we show that the condition is equivalent to a property of the representations of the group induced by the linear representations of R and C viz., that they have a single irreducible component in common, and neither induced representation contains this component more than once.

2. A Lemma on primitive idempotents.

LEMMA 1. *Let two subgroups R and C of a group G have representations of the first degree θ and ϕ respectively. If for any element $s \in G$ the condition*

$$s \in CR \rightleftarrows \theta(r) = \phi(c)$$

holds for every pair of elements $r \in R$, $c \in C$ for which $srs^{-1} = c$ then $e = PN$ is a multiple of a primitive idempotent, where

$$P = \sum_{r \epsilon R} r\, \theta(r), \quad N = \sum_{c \epsilon C} c\, \phi(c).$$

Proof. First note that

$$Pr_1\theta(r_1) = \sum_{r \epsilon R} r\, \theta(r)\, r_1\theta(r_1) = \sum_{r \epsilon R} rr_1\theta(rr_1) = P,$$

where r_1 is any element of R. Similarly $Nc_1\phi(c_1) = N$. Consider the expression $PNsPN$. If $s \in CR$, $s = cr$ say, then

$$PNsPN = PNcrPN = \theta^{-1}(r)\, \phi^{-1}(c)PNPN = \theta^{-1}(r)\, \phi^{-1}(c)(PN)^2.$$

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On the other hand if $s \notin CR$ then the condition of the lemma implies the existence of a pair $r \in R$ and $c \in C$ such that $srs^{-1} = c$ and $\theta(r) \neq \phi(c)$. In this case

$$PNsPN = \theta(r)\, PNsrs^{-1}\, sPN = \theta(r)\, PNcsPN = \theta(r)\, \phi^{-1}(c)\, PNsPN.$$

Hence:

$$PNsPN(1 - \theta(r)\, \phi^{-1}(c)) = 0.$$

Since $\theta(r) \neq \phi(c)$, we have $PNsPN = 0$. Writing $e = PN$ we get

$$eAe = \Lambda e^2 \tag{2.1}$$

where A is the group algebra of G over the field of representation Λ. Note that $e^2 \neq 0$, otherwise $eAeA = 0$ and eA is a nilpotent right ideal, whereas the group algebra is semi-simple. Also $eA \neq 0$ otherwise $e = 0$ which is impossible. In fact the coefficient of the unit element I in PN is $\sum\theta(\dot{r})\, \phi(\dot{c})$, and the summation is over all $\dot{r}$, $\dot{c}$ for which $\dot{r}\dot{c} = I$, i.e., over all $\dot{r} \in R \cap C$. Now $I\dot{r}I^{-1} = \dot{c}^{-1}$ and since $I \in CR$ the condition of the lemma gives $\theta(\dot{r}) = \phi^{-1}(\dot{c})$. Hence the coefficient of I in PN is $\sum\theta(\dot{r})\, \theta^{-1}(\dot{r}) = R \cap C{:}\,1 \neq 0$. By considering the expression $Ps^{-1}N$ and reasoning exactly as above we find that $PAN = \Lambda PN = \Lambda e$. Since $PAN \supset PNAPN = eAe$ we get

$$\Lambda e \supset eAe. \tag{2.2}$$

Combining equations (2.1) and (2.2) we have $e^2 = \lambda e$, so that e is a multiple of an idempotent. Besides it is seen from (1) that eAe is a field and hence by a well-known theorem e is a multiple of a primitive idempotent (**1**, p. 36).

3. A necessary and sufficient condition.

LEMMA 2. *The condition of Lemma 1 is satisfied if and only if the representations of G induced by the linear representations of R and C have one and only one irreducible component in common, and neither induced representation contains this component more than once.*

Proof. (a) First assume that the condition of Lemma 1 holds. Let

$$e_R = \frac{1}{R{:}\,1} P = \frac{1}{R{:}\,1} \sum r\, \theta(r),$$

$$e_C = \frac{1}{C{:}\,1} N = \frac{1}{C{:}\,1} \sum c\, \phi(c),$$

where $R{:}\,1$ and $C{:}\,1$ are the orders of the subgroups R and C respectively and the summations are taken over all $r \in R$ and $c \in C$. Since $Pr\, \theta(r) = P$ (proof of Lemma 1) we see that $P^2 = R{:}\,1$. so that $e_R^2 = e_R$. Similarly $e_C^2 = e_C$. Then e_R and e_C are primitive idempotents of the subalgebras over R and C respectively (**5**, p. 46). We have

$$e_R = \sum_j e^j, \quad e_C = \sum_j \bar{e}^j$$

where e^j, $\bar{e}^j$ are indempotents or 0 belonging to the jth Wedderburn component of A. Now because the condition of Lemma 1 holds:

$$\dim e_C A e_R = \dim \sum \bar{e}^j A\, e^j = 1,$$

therefore $\bar{e}^j A\, e^j = 0$ for all j except one, say $j = k$, and we have either $e^j = 0$ or $\bar{e}^j = 0$ if $j \neq k$. Moreover $\dim \bar{e}^k A e^k = 1$ so that e^k and $\bar{e}^k$ are primitive idempotents; hence the right ideals $e_R A$ and $e_C A$ which give the representations of G induced by the linear representations θ and ϕ of R and C respectively have a single minimal right ideal in common.

(b) Assume that the induced representations of G have only one component in common, each containing it with multiplicity one. Let

$$e_R = e + \ldots, \quad e_C = \bar{e} + \ldots,$$

where e and $\bar{e}$ are from the same Wedderburn component and the decompositions have no other component in common. We may suppose that $(e_R e_C)^2 \neq 0$ since under present assumptions this condition can always be secured by transforming the group C with a suitable element g of G. Thus:

$$(e_R g e_C g^{-1})^2 = eg\bar{e}g^{-1}eg\bar{e}g^{-1} = (eg\bar{e}g^{-1}e)\, g\bar{e}g^{-1}$$

and the last expression in brackets cannot vanish for all $g \in G$ otherwise:

$$0 = \sum_g eg\bar{e}g^{-1}e = e(\sum_g g\bar{e}g^{-1})\, e = \lambda e^2 \neq 0,$$

the final step arising from the fact that the bracketted expression is a central element of the subalgebra to which e belongs. Hence with suitable choice of g: $eg\bar{e}g^{-1}e \neq 0 \rightarrow eg\bar{e} \neq 0$ and since e is a primitive idempotent $eg\bar{e}g^{-1}e = \lambda_g\, e$. Returning to the first equation:

$$(e_R g e_C g^{-1})^2 = \lambda_g (eg\bar{e})\, g^{-1} \neq 0.$$

Now $0 \neq (e_R e_C)^2 = e\bar{e}e\bar{e}$, so that $e_R e_C$ is a multiple of a primitive idempotent. Moreover $e_C e_R = \bar{e}e \neq 0$. Also:

$$e_C\, s\, e_R = \bar{e}se = \mu_s \bar{e}e = \mu_s\, e_C\, e_R.$$

Because the last expression has only terms of the form cr, the same is true of the first. Therefore $s \notin CR \rightarrow \mu_s = 0$. On the other hand if $s \in CR$ then it is clear that $\mu_s \neq 0$. Let us suppose that $s \notin CR$ so that $e_C\, s\, e_R = 0$, i.e.:

$$\sum c\, s\, r\, \theta(r)\, \phi(c) = 0.$$

Consider terms of the form cs in the above; such exist, e.g. when $r = I$, the unit element. These terms occur only when $sr = c_r\, s$. Hence expressions with cs are

$$\sum{}'\, c\, c_r\, s\, \theta(r)\, \phi(c)$$

where the sum is now over c and such r for which $sr = c_r\, s$, i.e., for which $srs^{-1} = c_r$. We have then:

$$\sum{}' c\, c_r\, s\, \theta(r)\, \phi(c) = \sum{}' cs\, \theta(r)\, \phi(c)\, \phi^{-1}(c_r).$$

Now Lemma 1 required that for $s \notin CR$ there should exist r, c_r with $srs^{-1} = c_r$ and $\theta(r) \neq \phi(c_r)$; hence to negate this condition of the lemma we assume that the equality always holds and we get

$$(\sum c\, \phi(c))\, s = 0 \rightarrow e_C = 0.$$

which is impossible. Therefore the condition of Lemma 1 must be satisfied. We have now established Lemma 2.

4. Calculation of the character. The character of the representation corresponding to the idempotent derived from PN (Lemma 1) can be calculated by the formula

$$\chi(g) = \frac{n}{i(R \cap C: 1)} \sum \theta(r)\, \phi(c), \tag{4.1}$$

where χ is the character of the irreducible representation corresponding to the primitive idempotent formed from R and C; n is the degree of the irreducible representation; i is the index of the normalizer of the element g; r, c are elements of R and C and θ, ϕ are their respective signatures. The summation is taken over all r, c for which $rc \in \mathfrak{C}(g)$, the class of elements conjugate to g.

Proof of (4.1). In the first place,

$$\sum_{s \in G} s(PN)\, s^{-1}$$

is an element of the centre of the subalgebra *to which PN belongs*[1]; moreover the expression:

$$\sum_{t \in G} t\, \chi(t)$$

is the central idempotent of this subalgebra up to a multiple. Hence:

$$\lambda \sum_{t \in G} t\, \chi(t) = \sum_{s \in G} s(PN)\, s^{-1}.$$

Recalling that $PN = \sum rc\, \theta(r)\, \phi(c)$ and equating coefficients of g on both sides we get:

$$\lambda(\chi(g)) = \sum{}' \theta(r)\, \phi(c)$$

where the summation is over all r, c for which, for some s, $srcs^{-1} = g$. Now if this relation holds for a particular element s then it holds also for the element hs if h is an element of the normalizer $N(g)$ of g: $(hs)\, rc(hs)^{-1} = hgh^{-1} = g$. It follows that the contribution to the sum from each r, c for which $rc \in \mathfrak{C}(g)$ is repeated $N(g)$: 1 times. This permits us to write:

$$\lambda\chi(g) = (N(g):\ 1) \sum \theta(r)\, \phi(c), \tag{4.2}$$

[1] PN remains a multiple of a primitive idempotent even after extension of Λ to an algebraically closed field so that actually the centre of the Wedderburn component is of dimension 1.

the summation now being over all r, c such that $rc \in \mathfrak{C}(g)$. In particular if g is the identity element I of the group G then $N(g)\colon 1 = G\colon 1$ and $rc = I$ so that r and c must be from $R \cap C$ and the condition of Lemma 1 requires that $\theta(r) = \phi^{-1}(c)$. In consequence:

$$\lambda n = (G\colon 1)(R \cap C{:}1)$$

where $n = \chi(I)$ is the degree of the irreducible representation. Substitution for λ in (4.2) gives the result (4.1).

5. Application to GL(2, q). In the following paragraphs the preceding theory is used to find primitive idempotents of the group algebra of GL(2, q) as well as the actual bases for the corresponding irreducible representations. For this group there are **(7; 8)**

(a) $q - 1$ irreducible representations of degree 1,

(b) $q - 1$ irreducible representations of degree q,

(c) $\frac{1}{2}(q - 1)(q - 2)$ irreducible representations of degree $q + 1$,

(d) $\frac{1}{2}q(q - 1)$ irreducible representations of degree $q - 1$.

In each of the cases (a), (b), and (c) we find bases for the complete matrix algebra of the Wedderburn component. The writer has not been able to obtain similar results for the representations of (d) by the present method in the general case. For GL(2, 5) whose factor group with respect to its centre is S_5 the R and C subgroups for a representation of degree $q - 1 = 4$ can be obtained from the appropriate Young tableau for S_5.

6. Primitive idempotents of the group algebra of GL(2, q). We now obtain a pair of subgroups R and C of GL(2, q) which satisfy Lemma 1. By varying the signatures of R and C different primitive idempotents are obtained which will be classified in the next paragraph. The condition of Lemma 1 will be trivially satisfied if $R \cap C = I$ and $(R\colon 1)(C\colon 1) = G\colon 1$, for then $G = CR$ and

$$sRs^{-1} \cap C = crRr^{-1}c^{-1} \cap C = cRc^{-1} \cap C = I,$$

so that

$$\phi(sRs^{-1} \cap C) = 1 = \theta(R \cap s^{-1}Cs).$$

The order of GL(2, q) is $q(q - 1)(q^2 - 1)$; **(2)**. It is easy to find subgroups of orders $q(q - 1)$ and $q^2 - 1$ having only the identity I in common; take for R the triangular subgroup

$$R = \left\{\begin{pmatrix} \alpha & \beta \\ & 1 \end{pmatrix}\right\}$$

where α is any non-zero mark of GF(q) and β is any mark of this Galois field of

$q = p^n$ elements. Then $R: 1 = q(q-1)$. If ρ is a primitive element of $\mathrm{GF}(q)$ and $\alpha = \rho^a$ then

$$\theta\begin{pmatrix} \alpha & \beta \\ & 1 \end{pmatrix} = \epsilon^a,$$

with ϵ a root of $x^{q-1} = 1$ in the field of the representation, the field of complex numbers say, gives a representation of R of the first degree. Each root of this equation gives a distinct linear representation and we get them all in this way since

$$\left\{\begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}\right\}$$

is the commutator of R and its index in the latter is $q - 1$.

For the subgroup C of order $C: 1 = q^2 - 1$ we take the cyclic group generated by an element of $\mathrm{GL}(2, q)$ similar to

$$\begin{pmatrix} \sigma & \\ & \sigma^q \end{pmatrix}$$

in which σ is a primitive root of the quadratic extension field $\mathrm{GF}(q^2)$. That is

$$C = \left\{T\begin{pmatrix} \sigma & \\ & \sigma^q \end{pmatrix}^m T^{-1}\right\} \tag{6.1}$$

where T is chosen so that the elements lie in $\mathrm{GL}(2, q)$. Now

$$\phi\left(T\begin{pmatrix} \sigma^m & \\ & \sigma^{mq} \end{pmatrix} T^{-1}\right) = \omega^m$$

where ω is a root of the equation $x^{q^2-1} = 1$ can clearly give all $q^2 - 1$ linear representations of the cyclic group C. Recalling the definition of P and N (Lemma 1) we see that

$$PN = \sum_{\beta} \sum_{a=1}^{q-1} \sum_{m=1}^{q^2-1} \begin{pmatrix} \rho^a & \beta \\ & 1 \end{pmatrix} T\begin{pmatrix} \sigma^m & \\ & \sigma^{mq} \end{pmatrix} T^{-1} \epsilon^a \omega^m$$

is a multiple of a primitive indempotent for each choice of ϵ and ω.

7. Classification of the primitive idempotents. The primitive idempotents of the preceding section can be distinguished through the values of the corresponding irreducible characters on a suitable element of $\mathrm{GL}(2, q)$. We use for the calculation the formula (4.1).

Let us calculate $\chi(g_1)$ for

$$g_1 = \begin{pmatrix} \rho^{a_1} & \\ & \rho^{b_1} \end{pmatrix}, \qquad a_1 \neq b_1.$$

Here $N(g): 1 = q(q+1)$. Recall that $R \cap C: 1 = 1$. A simple choice for T in (6.1) can be obtained by assuming a matrix with unknown coefficients

and then determining them so as to ensure that (6.1) lie in GL$(2, q)$. We shall use

$$T = \begin{pmatrix} 1 & 1 \\ \sigma & \sigma \end{pmatrix}.$$

With this choice of T we have, for C,

$$c = \left\{ \begin{pmatrix} (\sigma^{m+q} - \sigma^{mq+1})\,\Delta^{-1} & (\sigma^{mq} - \sigma^{m})\,\Delta^{-1} \\ -\sigma^{q+1}(\sigma^{mq} - \sigma^{m})\,\Delta^{-1} & (\sigma^{(m+1)q} - \sigma^{m+1})\,\Delta^{-1} \end{pmatrix} \right\}, \qquad \Delta = \sigma^{q} - \sigma,$$

so that

$$rc = \begin{pmatrix} \alpha & \beta \\ & 1 \end{pmatrix} \begin{pmatrix} (\sigma^{m+q} - \sigma^{mq+1})\,\Delta^{-1} & (\sigma^{mq} - \sigma^{m})\,\Delta^{-1} \\ -\sigma^{q+1}(\sigma^{mq} - \sigma^{m})\,\Delta^{-1} & (\sigma^{(m+1)q} - \sigma^{m+1})\,\Delta^{-1} \end{pmatrix}.$$

Since g_1 has two distinct eigenvalues the requirement that $rc \in \mathfrak{C}(g_1)$ will be satisfied if we make sure that trace (rc) = trace g_1 and that determinant (rc) = determinant g_1. These conditions yield

$$\text{(7.1)} \qquad \begin{aligned} \alpha(\sigma^{m+q} - \sigma^{mq+1}) - \sigma^{q+1}(\sigma^{mq} - \sigma^{m})\,\beta + \sigma^{(m+1)q} - \sigma^{m+1} &= (\rho^{a_1} + \rho^{b_1})\Delta, \\ \alpha\,\sigma^{m(q+1)} &= \rho^{a_1+b_1}. \end{aligned}$$

For fixed m the latter equation determines α: $\alpha = \rho^{a_1+b_1-m}$. Here σ and ρ have been so related that $\sigma^{q+1} = \rho$. Now β is uniquely determined by the former equation if $\sigma^{m} - \sigma^{mq} \neq 0$. In this case r is fully determined for a given c and

$$\phi(c) = \omega^{m}, \quad \theta(r) = \epsilon^{a_1+b_1-m}.$$

On the other hand if $\sigma^{m} - \sigma^{mq} = 0$ then β may be any of the q marks of GF(q); also $m = t(q+1)$ where $1 < t < q - 1$. Since $\sigma^{q+1} = \rho$ and $\rho^{q} = \rho$, the first equation of (7.1) gives

$$\rho^{a_1+b_1-2t}\,\rho^{t} + \rho^{t} = \rho^{a_1} + \rho^{b_1},$$

and after simplification

$$(\rho^{t})^2 - (\rho^{a_1} + \rho^{b_1})\,\rho^{t} + \rho^{a_1+b_1} = 0,$$

so that either $t = a_1$ or $t = b_1$. There are then just two possibilities for the element c determined by $m = a_1(q+1)$ and by $m = b_1(q+1)$. Each of these determines q possibilities for the element r. Also each value of m fixes the signature θ of the corresponding elements r through the second equation of (7.1), so that for one case

$$\phi(c) = \omega^{a_1(q+1)}, \quad \theta(r) = \epsilon^{b_1-a_1},$$

and for the other

$$\phi(c) = \omega^{b_1(q+1)}, \quad \theta(r) = \epsilon^{a_1-b_1}.$$

We are now able to write:

$$\sum \theta(r)\,\phi(c) = \sum_{m=1}^{q^2-1} \epsilon^{a_1+b_1-m}\,\omega^{m} + q[\epsilon^{b_1-a_1}\,\omega^{a_1(q+1)} + \epsilon^{a_1-b_1}\,\omega^{b_1(q+1)}];$$

equation (4.1) now gives:

$$(7.2)\quad \chi(g_1) = \frac{n}{q(q+1)}\left\{ \sum_{m=1}^{q^2-1} \epsilon^{a_1+b_1-m}\omega^m + q(\epsilon^{b_1-a_1}\omega^{a_1(q+1)} + \epsilon^{a_1-b_1}\omega^{b_1(q+1)}) \right\},$$

where $m \not\equiv 0 (\mathrm{mod}\ q+1)$. The following cases can be considered:

Case I: $\omega = \epsilon$. Then $\chi(g_1) = n\epsilon^{a_1+b_1}$ and each of the $q-1$ roots ϵ of the equation $x^{q-1} = 1$ gives rise to a distinct character of degree n. Since g_1 is not a central element we know that $n = 1$ and that these are the $q-1$ linear characters **(7)**.

Case II: $$(\omega/\epsilon)^{q+1} = 1, \quad \omega \neq \epsilon.$$

Now $\chi(g_1) = (n/q)\,\epsilon^{a_1+b_1}$ and we have $q-1$ distinct characters, one for each ϵ. Their degree is $n = q$. We remark that each choice of ϵ gives a distinct character of degree q but that for fixed ϵ, ω can take q values. In this way we get q distinct idempotents associated with each irreducible representation of degree q. This remark will be useful in the next section.

Case III: $$(\omega/\epsilon)^{q+1} \neq 1.$$

In this case the summation term in equation (7.2) above is zero and we have:

$$\chi(g_1) = \frac{n}{q+1}(\epsilon^{b_1-a_1}\omega^{a_1(q+1)} + \epsilon^{a_1-b_1}\omega^{b_1(q+1)}).$$

Writing $\omega^{q+1} = \epsilon/\epsilon_1$, where ϵ_1 is a different root of the equation $x^{q-1} = 1$, we get finally:

$$\chi(g_1) = \frac{n}{q+1}(\epsilon^{b_1}\epsilon_1^{-a_1} + \epsilon^{a_1}\epsilon_1^{-b_1}).$$

In this formula ϵ can take $q-1$ values, to each of which ϵ_1 may take $q-2$ values, since $\epsilon_1 \neq \epsilon$. Hence values may be assigned to both in $(q-1)(q-2)$ ways; however, half of these lead to the same character as the other half. The results indicate that these are the $\frac{1}{2}(q-1)(q-2)$ irreducible characters of degree $q+1$. We note that ϵ and ϵ_1 fix the character but that ω is free to take $q+1$ values, giving rise to $q+1$ distinct idempotents belonging to the same irreducible representation of degree $q+1$.

We have now obtained, to within a multiple λ, primitive idempotents for all the irreducible representations of degrees 1, q, and $q+1$. The idempotents themselves can be determined since the trace $\chi_R(\lambda PN)$ in the regular representation is equal to the degree of the irreducible representation.

The irreducible representations of degree $q-1$ have not appeared. The writer has been unable to find them by other choices of R and C which have merely led to one or other of the representations already obtained.

8. Bases for the irreducible subalgebras. For the linear representations there is nothing to be discussed as each idempotent is already a basis and the linear characters are in fact representations.

Recalling Cases II and III of the previous section we see that for each of the irreducible representations of degree q or $q+1$ there are as many distinct primitive idempotents as the degree n; these, together with an equal number of primitive idempotents obtained by reversing the roles of R and C, will be used to construct the n^2 basis elements of the matrix subalgebra.

We notice that in both cases II and III the signature ϵ remains fixed for all the equivalent idempotents; the changes in ω distinguish them. Thus the terms P in

$$\dot{e}_i = \lambda P N_i$$

are the same[3] for all the idempotents $\dot{e}_i$. The N_i stand for N under the different choices of ω. Now:

$$\dot{e}_i\dot{e}_j = \lambda^2 PN_iPN_j = \mu\lambda^2 PN_j = \nu\dot{e}_j.$$

The second step is from the fact that $PAN = \Lambda PN$ (§2). Thus

$$\dot{e}_i\dot{e}_i\dot{e}_j = \nu\dot{e}_i\dot{e}_j = \nu^2\dot{e}_j = \dot{e}_i\dot{e}_j = \nu\dot{e}_j$$

and hence

$$(\nu^2-\nu)\,\dot{e}_j = 0,$$

implying that either $\nu=0$ or $\nu=1$, and so $\dot{e}_i\dot{e}_j=\dot{e}_j$ or 0. Similarly $\dot{e}_j\dot{e}_i=\dot{e}_i$ or 0.

LEMMA 3. $$\dot{e}_i\dot{e}_j = \dot{e}_j \rightleftarrows \dot{e}_j\dot{e}_i = \dot{e}_i.$$

Proof. If A is the group algebra then

$$\dot{e}_i\dot{e}_jA = \dot{e}_jA \subset \dot{e}_iA$$

and, since $\dot{e}_iA =$ is minimal, $\dot{e}_jA = \dot{e}_iA \to \dot{e}_i = \dot{e}_jx$, so that

$$\dot{e}_j\dot{e}_i = \dot{e}_j\dot{e}_jx = \dot{e}_jx = \dot{e}_i.$$

COROLLARY. $$\dot{e}_i\dot{e}_j = 0 \rightleftarrows \dot{e}_j\dot{e}_i = 0.$$

However this is not possible; for let $\dot{e}_i\dot{e}_j = 0 = \dot{e}_j\dot{e}_i$, then $\dot{e}_i$, $\dot{e}_j$ are primitive mutually orthogonal idempotents of a matrix algebra: we identify them with, say, e_{11} and e_{22}. Then

$$\dot{e}_iA\dot{e}_j = e_{11}Ae_{22} = \Lambda e_{12} \neq 0;$$

but

$$\dot{e}_iA\dot{e}_j = \lambda^2 PN_iAPN_j = \Lambda PN_j = \Lambda\dot{e}_j,$$

so that $e_{12} = \tau\dot{e}_j$ and this is impossible. Hence

(8.1) $$\dot{e}_i\dot{e}_j = \dot{e}_j.$$

Now we interchange R and C, i.e., the group formerly taken for C will be used for R and vice versa. In terms of the original R and C the idempotents are now:

$$e = \lambda NP.$$

[3] λ also remains the same since the coefficient of I in PN_i is 1, so that $\chi_R(\dot{e}_i) = \lambda(G:1)$.

Since the trace and determinant of an element cr is the same as that of the element rc the equations (7.1) are not changed and the features of cases II and III remain the same. Let $e_1, e_2, \ldots e_j, \ldots$ be the new system of equivalent idempotents belonging to a particular irreducible representation of degree q or $q + 1$, so that $e_i = \lambda N_i P$. As for the $\hat{e}_i$, we prove for e_i in an entirely analogous way:

$$e_i e_j = e_i. \tag{8.2}$$

LEMMA 4. *For the systems $\hat{e}_i$ and e_i of a particular representation:*

$$\begin{aligned} \hat{e}_i e_j &= 0, && i \neq j, \\ \hat{e}_i e_i &\neq 0, && \\ e_i \hat{e}_j &\neq 0, && \text{for all } i, j. \end{aligned} \tag{8.3}$$

Proof. In the first place $N_i N_j$ must vanish since N_i, N_j are multiples of different smallest central idempotents of the group algebra of C. Thus $\hat{e}_i e_j = \lambda^2 P N_i N_j P = 0$. On the other hand

$$\hat{e}_i e_i = \lambda^2 P N_i N_i P = \lambda^2 (C\colon 1)\, P N_i P \neq 0,$$

otherwise on right multiplication by N_i we would get

$$\lambda^2 P N_i P N_i = \hat{e}_i^2 = \hat{e}_i = 0.$$

Moreover,

$$e_i \hat{e}_j = \lambda^2 N_i P P N_j = \lambda^2 (R\colon 1)\, N_i P N_j \neq 0,$$

otherwise on left multiplication by P we should get

$$\lambda^2 P N_i P N_j = 0 = \hat{e}_i \hat{e}_j = \hat{e}_j.$$

Relations (8.3) show $\hat{e}_i$ and e_j are distinct; for if $\hat{e}_i = e$

$$\hat{e}_i = \hat{e}_i \hat{e}_i = \hat{e}_i e_j = 0.$$

Again if $\hat{e}_i = e_i$ then

$$\hat{e}_i = \hat{e}_j \hat{e}_i = \hat{e}_j e_i = 0.$$

LEMMA 5. *A matrix basis for the irreducible subalgebra, corresponding to a particular idempotent of degree q or $q + 1$, is given by $E_{ij} = e_i \hat{e}_j$ with suitable normalization.*[3]

Proof.

$$\begin{aligned} E_{ij} E_{km} &= e_i \hat{e}_j e_k \hat{e}_m = 0, && \text{if } j \neq k. \\ E_{ij} E_{jm} &= \lambda^4 e_i P N_j N_j P P N_m = \lambda^3 (C\colon 1)(R\colon 1)\, e_i P N_j P N_m && \\ &= \lambda (C\colon 1)(R\colon 1)\, e_i \hat{e}_j \hat{e}_m = \lambda (C\colon 1)(R\colon 1)\, e_i \hat{e}_m && \\ &= \lambda (C\colon 1)(R\colon 1)\, E_{im}. && \end{aligned}$$

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[3]The referee has kindly drawn the author's attention to an interesting paper by Frame (3) in which a pair of subgroups is used to give an irreducible representation of the group.

The normalized system is thus $\bar{E}_{ij} = E_{ij}/\lambda(C: 1)(R: 1)$, for then

$$\bar{E}_{ij}\bar{E}_{km} = 0, \qquad j \neq k,$$

and $\bar{E}_{ij}\bar{E}_{jk} = \bar{E}_{ik}$. The λ is known from the regular trace. The $\bar{E}_{ii}$ are primitive idempotents since

$$\bar{E}_{ii}A\bar{E}_{ii} = e_i\bar{e}_iAe_i\bar{e}_i = \Lambda e_i\bar{e}_i = \Lambda\bar{E}_{ii},$$

the second step arising from the fact that e_i is a primitive idempotent. The $\bar{E}_{ij}$ give bases for the actual construction of any of the irreducible representations of degree q or $q + 1$.

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NOTE ON THE ALGEBRA OF S-FUNCTIONS

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Considerable advance has been made recently towards a systematic method of evaluating the "product" $\{\mu\} \otimes \{\lambda\}$, most notably in the methods of Robinson **(3)**, Littlewood **(2)**, and Todd **(5)** and the differential operator technique of H. O. Foulkes.

In this note a formula is derived which expresses $\{\mu\} \otimes \{2r\}$ in terms of products $\{\mu\} \otimes \{n\}$ (where $n < 2r$) and the more easily calculated functions $\{\mu\} \otimes S_N$ **(1**, p. 235**)**.

The products $\{\mu\} \otimes \{r\}$, where (r) denotes the partition of r consisting of a single element, are of particular interest because of their applications in invariant theory.

For brevity we will denote $\{\mu\} \otimes S_i$ by t_i. Then we have **(4**, p. 374**)**

$$\{\mu\} \otimes \{r\} = \sum_{(\alpha)} \frac{1}{\alpha_1! \ldots \alpha_r!} \left(\frac{t_1}{1}\right)^{\alpha_1} \ldots \left(\frac{t_r}{r}\right)^{\alpha_r}.$$

Hence, if we define $\{\mu\} \otimes \{0\} = 1$, we have

$$\prod_{i=1}^{\infty} \exp\left(\frac{t_i}{i} z^i\right) = \sum_{r=0}^{\infty} \{\mu\} \otimes \{r\}\, z^r.$$

Now

$$\prod_{i=1}^{\infty} \exp\left(\frac{t_i}{i} (+z)^i\right) \cdot \prod_{i=1}^{\infty} \exp\left(\frac{t_i}{i} (-z)^i\right) = \prod_{i=1}^{\infty} \exp\left(\frac{t_{2i}}{i} z^{2i}\right),$$

i.e.,

$$\sum_{r=0}^{\infty} \{\mu\} \otimes \{r\}\, z^r \cdot \sum_{r=0}^{\infty} \{\mu\} \otimes \{r\}\, (-z)^r = \prod_{i=1}^{\infty} \exp\left(\frac{t_{2i}}{i} z^{2i}\right).$$

Equating coefficients of z^{2k} and transposing, we have:

$$\begin{aligned}\{\mu\} \otimes \{2k\} = (\{\mu\} \otimes \{2k-1\})(\{\mu\}) - (\{\mu\} \otimes \{2k-2\})(\{\mu\} \otimes \{2\}) \\ + \ldots + \frac{(-1)^{k+1}}{2} (\{\mu\} \otimes \{k\})^2 + \tfrac{1}{2} \sum_{(\beta)} \frac{1}{\beta_1! \ldots \beta_k!} \left(\frac{t_2}{1}\right)^{\beta_1} \ldots \left(\frac{t_{2k}}{k}\right)^{\beta_k}\end{aligned}$$

This formula is particularly useful in calculating $\{m\} \otimes \{4\}$, since

$$\{m\} \otimes \{4\} = (\{m\} \otimes \{3\})(\{m\}) - \tfrac{1}{2}(\{m\} \otimes \{2\})^2 + \tfrac{1}{2}(\tfrac{1}{2}t_2^2 + t_4)$$

and explicit formulas **(4**, pp. 380-382**)** are available for $\{m\} \otimes \{3\}$ and $\{m\} \otimes \{2\}$.

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SOME REMARKS ON THE CHARACTERS OF THE SYMMETRIC GROUP, II

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Introduction. Let p be a fixed prime number. We denote by $k(n)$ the number of partitions of n. As is well known, the number of ordinary irreducible characters of the symmetric group S_n is $k(n)$. We set $k(0) = 1$ and

$$(1)\qquad l(b) = \sum_{b_0,\dots,b_{p-1}} k(b_0)\, k(b_1) \dots k(b_{p-1}) \qquad \left(\sum_{i=0}^{p-1} b_i = b,\ 0 \leqslant b_i \leqslant b\right),$$

$$(2)\qquad l^*(b) = \sum_{b_1,\dots,b_{p-1}} k(b_1)\, k(b_2) \dots k(b_{p-1}) \qquad \left(\sum_{i=1}^{p-1} b_i = b,\ 0 \leqslant b_i \leqslant b\right).$$

Two ordinary irreducible representations of S_n belong to the same p-block if and only if they have the same p-core (**10**; **2**; **11**). The number of ordinary irreducible characters belonging to a p-block of weight b is independent of the p-core and is equal to $l(b)$ (**16**; **12**; also **11**; **15**). This may be also easily proved by applying the theory of p-quotients (**6**; **4**). Moreover we have the following theorem (**13**; also **4a**; **8**; **15**; **16**).

THEOREM 1. *The number of modular irreducible characters belonging to a p-block of weight b is $l^*(b)$.*

In the present paper we shall give a simple proof for this theorem. We shall then derive some new properties of decomposition numbers of S_n.

1. We denote by χ_α the character of the irreducible representation $[\alpha]$ corresponding to a Young diagram $[\alpha]$. We set $r(\alpha, \alpha') = (-1)^s$ if a diagram $[\alpha']$ of S_{n-g} is obtained from $[\alpha]$ by removing a g-hook of leg length s. Otherwise we set $r(\alpha, \alpha') = 0$. Then the Murnaghan-Nakayama recursion formula (**7**; **9**) is expressed as follows:

If G is an element of S_n containing a g-cycle P and $\bar{G}$ is the permutation of $n - g$ symbols arising from G by removing this cycle, then

$$(3)\qquad \chi_\alpha(G) = \sum_{\alpha'} r(\alpha, \alpha')\, \chi_{\alpha'}(\bar{G}),$$

where $[\alpha']$ ranges over all diagrams of S_{n-g}.

If $[\alpha]$ is a diagram with p-core $[\alpha_0]$ then the summation in (3) may be limited to those $[\alpha']$ with the same p-core $[\alpha_0]$.

We set $n = n' + tp$ $(0 \leqslant n' < p)$ and consider an element G of S_n such that

$$G = W \cdot Q_1 \cdot Q_2 \dots Q_t,$$

where no two of Q_i have common symbols and each Q_i is a cycle of length

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$a_i p$ $(a_1 > a_2 > \ldots > a_s)$ and where W is any permutation on the fixed symbols of $P = Q_1 . Q_2 \ldots Q_s$. We set

$$a = \sum_i a_i \qquad (0 < a < t).$$

Then P is called an element of *type* $(a_1, a_2, \ldots, a_s)$ and of *weight* a. The number of elements of weight a such that they all lie in different conjugate classes of S_n is $k(a)$. If we set

$$\sum_{a=0}^{t} k(a) = r, \tag{4}$$

then we have a system of elements of weight a $(a = 0, 1, 2, \ldots, t)$

$$P_0 = 1, P_1, \ldots, P_{r-1},$$

such that they all lie in different conjugate classes of S_n and every element of weight a $(0 < a < t)$ is conjugate to one of them. Every conjugate class contains an element of the form VP_i, where i is uniquely determined by the class and where V is a p-regular element of S_{n-ap}, if P_i is of weight a. Since the number $k^*(n)$ of modular irreducible representations of S_n is equal to the number of p-regular classes of S_n, we have

$$k(n) = \sum_{a=0}^{t} k^*(n - ap)\, k(a). \tag{5}$$

Let P_i be an element of type $(a_1, a_2, \ldots, a_s)$ and of weight a. Let $[\alpha_0]$ be a p-core with m nodes and $n = m + bp$. Then the number of diagrams of S_{m+jp} with p-core $[\alpha_0]$ is $l(j)$. We denote by $\chi_\beta{}^{(a)}$ the character of the irreducible representation $[\beta]$ of S_{n-ap} corresponding to a diagram $[\beta]$. Let us denote by B the block of S_n with p-core $[\alpha_0]$. Applying the Murnaghan-Nakayama recursion formula iterated s times to $[\alpha] \subset B$, we obtain

$$\chi_\alpha(VP_i) = \begin{cases} \sum_\beta h(\alpha, \beta)\, \chi_\beta^{(a)}(V),\ [\beta] \subset B^{(a)} & (\text{for } a < b), \\ 0 & (\text{for } b < a), \end{cases} \tag{6}$$

where the $h(\alpha, \beta)$ are rational integers and $B^{(a)}$ denotes the block of S_{n-ap} with p-core $[\alpha_0]$. If $a < b$ then $B^{(a)}$ is of weight $b - a$. Let $\phi_\lambda{}^{(a)}$ be the character of S_{n-ap} in the modular irreducible representation λ. We then have

$$\chi_\beta^{(a)}(V) = \sum_\lambda d_{\beta\lambda}^{(a)} \phi_\lambda^{(a)}(V) \qquad (V \text{ in } S_{n-ap},\ p\text{-regular}), \tag{7}$$

where the $d_{\beta\lambda}{}^{(a)}$ are the decomposition numbers **(1)** of S_{n-ap}. Hence (6), combined with (7), yields

$$\chi_\alpha(VP_i) = \sum_\lambda u_{\alpha\lambda}{}^i \phi_\lambda^{(a)}(V), \tag{8}$$

where the $u_{\alpha\lambda}{}^i$ are rational integers. If $b < a$ then $u_{\alpha\lambda}{}^i = 0$ for every λ, and if $a < b$ then $u_{\alpha\lambda}{}^i = 0$ for $\lambda \not\subset B^{(a)}$. Let $D = (d_{\alpha\lambda})$ be the decomposition matrix of S_n. Then

$$\chi_\alpha(V) = \sum_\lambda d_{\alpha\lambda}\phi_\lambda(V) \qquad (V \text{ in } S_n,\ p\text{-regular}). \tag{9}$$

Hence, for $P_0 = 1$, we have

(10) $$u_{\alpha\lambda}^{\;0} = d_{\alpha\lambda}.$$

We arrange these numbers $u_{\alpha\lambda}^{\;i}$ for a fixed i in the form of a matrix

(11) $$U^i = (u_{\alpha\lambda}^{\;i}),$$

with α as row index and λ as column index, and set

(12) $$U = (U^0, U^1, \ldots, U^{t-1}).$$

Each column of U is given by a pair (i, λ). It follows from (5) that the number of such columns is $k(n)$ (note that the number of elements P_i of weight a is $k(a)$), whence U is a square matrix of the same degree as the matrix $Z = (\chi_\alpha(G))$ of the group characters χ_α of S_n. According to (8) we have the formula

(13) $$Z = UA.$$

Here A is a square matrix such that

(14) $$A = \begin{bmatrix} \Phi^{(0)} & & & & 0 \\ & \Phi^{(1)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \Phi^{(t)} \end{bmatrix},$$

where, for each a, the matrix $\Phi^{(a)} = (\phi_\lambda^{(a)}(V))$ of the modular group characters of S_{n-ap} appears in the main diagonal with multiplicity $k(a)$ if the rows and columns are arranged suitably. Since Z is non-singular, so is U:

(15) $$|U| \neq 0.$$

Proof of Theorem 1. It follows from (8) that, if the rows and columns of U are taken in a suitable order, U breaks up completely into q matrices $U_1, U_2, \ldots, U_q$, each U_k corresponding to a block B_k of S_n. Denote by x_k the number of ordinary irreducible characters in B_k. It follows from $|U| \neq 0$ that each U-matrix U_k of B_k must necessarily be a square matrix of degree x_k and $|U_k| \neq 0$. Let B_k be a block of weight b with p-core $[\alpha_0]$. We then have $x_k = l(b)$. Denote by $f(a)$ the number of modular irreducible characters in a block of weight a with p-core $[\alpha_0]$. Since U_k is a square matrix of degree $l(b)$ we have by (8)

(16) $$l(b) = \sum_{a=0}^{b} f(a)\, k(b-a).$$

Since $l^*(0) = f(0) = 1$ and $l^*(1) = f(1) = p - 1$, we shall assume that $l^*(a) = f(a)$ for $a < b$. We then have by (**12**; Lemma 1)

$$f(b) = l(b) - \sum_{a=0}^{b-1} f(a)\, k(b-a)$$
$$= l(b) - \sum_{a=0}^{b-1} l^*(a)\, k(b-a) = l^*(b).$$

This completes the proof.

2. In what follows we shall be concerned with representations belonging to a fixed block B_k of weight b, so we may drop the subscript k. Applying (8) to the orthogonality relations

$$\sum_\alpha \chi_\alpha(VP_i)\,\chi_\alpha(V'P_j) = 0 \qquad (i \neq j),$$

we obtain

$$\sum_\alpha \chi_\alpha(VP_i)\,\chi_\alpha(V'P_j) = 0 \qquad [\alpha] \subset B, \quad (i \neq j), \tag{17}$$

whence

$$\sum_\alpha u_{\alpha\lambda}{}^i \chi_\alpha(V'P_j) = 0 \qquad [\alpha] \subset B, \quad (i \neq j). \tag{18}$$

We then have

$$\sum_\alpha u_{\alpha\lambda}{}^i u_{\alpha\kappa}{}^j = 0 \qquad [\alpha] \subset B, \quad (i \neq j). \tag{19}$$

For $P_j = P_0 = 1$, it follows from (18) that

$$\sum_\alpha u_{\alpha\lambda}{}^i \chi_\alpha(V) = 0 \qquad [\alpha] \subset B, \quad (i \neq 0), \tag{20}$$

where V is any p-regular element of S_n. Hence

$$\sum_\alpha u_{\alpha\lambda}{}^i d_{\alpha\kappa} = 0 \qquad [\alpha] \subset B, \quad (i \neq 0). \tag{21}$$

Since the U-matrix U_k of B is non-singular the identities (21) are linearly independent. Moreover the number of identities (21) is $l(b) - l^*(b)$ and hence the system of linearly independent identities (21) satisfied by the rows of the decomposition matrix D_k of B is complete.

We shall denote by $n(G)$ the order of the normalizer $N(G)$ of G in S_n. Applying (8) to the orthogonality relations

$$\sum_\alpha \chi_\alpha(VP_i)\,\chi_\alpha(VP_i) = n(VP_i),$$

we have

$$\sum_\lambda \Big(\sum_\alpha u_{\alpha\lambda}{}^i \chi_\alpha(VP_i)\Big)\,\phi_\lambda^{(a)}(V) = n(VP_i).$$

Let $\eta_\lambda{}^{(a)}$ be the character of the indecomposable constituent of the regular representation of S_{n-ap} which corresponds to $\phi_\lambda{}^{(a)}$. Then we have the character relation

$$\sum_\lambda \eta_\lambda^{(a)}(V)\,\phi_\lambda^{(a)}(V) = n^{(a)}(V),$$

where $n^{(a)}(V)$ denotes the order of the normalizer of V in S_{n-ap}. Hence

$$\sum_\alpha u_{\alpha\lambda}{}^i \chi_\alpha(VP_i) = \frac{n(VP_i)}{n^{(a)}(V)}\,\eta_\lambda^{(a)}(V), \qquad [\alpha] \subset B. \tag{22}$$

If P_i is an element of weight a with $n - ap$ 1-cycles, k_1 p-cycles, k_2 $2p$-cycles, $\ldots$, k_m mp-cycles, then (22) yields

$$\sum_{\alpha} u_{\alpha\lambda}{}^{i} u_{\alpha\kappa}{}^{i} = \frac{n(VP_i)}{n^{(a)}(V)} c_{\lambda\kappa}^{(a)} \tag{23}$$
$$= c_{\lambda\kappa}^{(a)} \prod_i (k_i!(ip)^{k_i}), \qquad [\alpha] \subset B,$$

where the $c_{\lambda\kappa}{}^{(a)}$ denote the Cartan invariants of S_{n-ap}.

3. Let $[\alpha]$ with p-core $[\alpha_0]$ belong to a block B of weight b and let $[\alpha]^*$ be its star diagram (**14**; also **4**; **11**; **17**). We shall write

$$[\alpha]^* = [\nu_0]\cdot[\nu_1]\cdot \ldots \cdot[\nu_{p-1}],$$

where the $[\nu_r]$ are the disjoint right constituents of $[\alpha]^*$. We assume that $[\nu_r]$ contains b_r nodes, where

$$b = b_0 + b_1 + \ldots + b_{p-1}, \tag{24}$$

and r is the leg length of the p-hook represented by its upper left-hand corner node. We denote by χ_{α^*} the character of (reducible) representation $[\alpha]^*$ of S_b corresponding to the star diagram $[\alpha]^*$ and by $f_\alpha{}^*$ its degree. Then

$$f_\alpha^* = \frac{b!}{b_0!\, b_1! \ldots b_{p-1}!} f_{\nu_0} f_{\nu_1} \ldots f_{\nu_{p-1}}, \tag{25}$$

where f_{ν_r} denotes the degree of the ordinary irreducible representation $[\nu_r]$ of S_{b_r} (**14**).

If P_b represents the product of b cycles, each of length p, on the last bp of n symbols, then P_b is of weight b and of type $(1, 1, \ldots, 1)$. Denote by $N(P_b)$ the normalizer of P_b in S_n. We then have $N(P_b) = \mathfrak{G}_1 \times \mathfrak{G}_2$, where $\mathfrak{G}_1$ is the subgroup of S_n which permutes only the first $n - bp$ symbols and which may be identified with S_{n-bp}. On the other hand

$$\mathfrak{G}_2 = S_b^*\mathfrak{Q}, \quad S_b^* \cap \mathfrak{Q} = 1, \tag{26}$$

where $\mathfrak{Q}$ is the subgroup generated by the b individual cycles of length p of P_b and is the normal subgroup of $\mathfrak{G}_2$, and $S_b{}^*$ is the subgroup of permutations which permute the cycles of P_b amongst themselves. We see that $S_b{}^*$ is isomorphic to the symmetric group S_b of b symbols. We denote by W the element of S_b which corresponds to W^* of $S_b{}^*$. The transitive subgroup $\mathfrak{G}_2$ of S_n is called the *generalized symmetric group* and is denoted by $S(b, p)$. The order of $S(b, p)$ is $b!p^b$. It may be verified that there are $l(b)$ conjugate classes of $S(b, p)$. For example we shall determine the conjugate classes of $S(2, 3)$. We set

$$Q_1 = (1\ 2\ 3), \qquad Q_2 = (4\ 5\ 6).$$

Then there exist two conjugate classes which are represented by

$$W_0{}^* = 1, \qquad W_1{}^* = (1\ 4)(2\ 5)(3\ 6).$$

A complete system of representatives for the conjugate classes of $S(2, 3)$ is given by

$$W_0^*,\ W_1^*,\ Q_1,\ Q_1^2,\ Q_1Q_2,\ Q_1Q_2^2,\ Q_1^2Q_2^2,\ W_1^*Q_1,\ W_1^*Q_1^2.$$

Each element is associated with a star diagram with 2 nodes by the following way:

$W_0^* = 1$		$[1^2]\cdot[0]\cdot[0]$
$W_1^* =$	$(1\ 4)(2\ 5)(3\ 6)$	$[2]\cdot[0]\cdot[0]$
$Q_1Q_2 =$	$(1\ 2\ 3)(4\ 5\ 6)$	$[0]\cdot[1^2]\cdot[0]$
$W_1^*Q_1 =$	$(1\ 4\ 2\ 5\ 3\ 6)$	$[0]\cdot[2]\cdot[0]$
$Q_1^2Q_2^2 =$	$(1\ 3\ 2)(4\ 6\ 5)$	$[0]\cdot[0]\cdot[1^2]$
$W_1^*Q_1^2 =$	$(1\ 4\ 3\ 6\ 2\ 5)$	$[0]\cdot[0]\cdot[2]$
$Q_1 =$	$(1\ 2\ 3)$	$[1]\cdot[1]\cdot[0]$
$Q_1^2 =$	$(1\ 3\ 2)$	$[1]\cdot[0]\cdot[1]$
$Q_1Q_2^2 =$	$(1\ 2\ 3)(4\ 6\ 5)$	$[0]\cdot[1]\cdot[1]$.

By the same way each conjugate class of $S(b, p)$ is uniquely associated with a star diagram with b nodes. Every conjugate class of $S(b, p)$ associated with $[\alpha]^*$ such that $[\nu_0] = [0]$ contains the elements of weight b. But the converse is not valid generally.

THEOREM 2. *The number of ordinary irreducible representations of $S(b, p)$ is $l(b)$ and there is a* (1-1) *correspondence between ordinary irreducible representations of $S(b, p)$ and star diagrams $[\alpha]^*$ containing b nodes.*

This, together with related theorems, will be proved in a forthcoming paper **(13a)**.

We denote by ζ_{α^*} the ordinary irreducible characters of $S(b, p)$ corresponding to a star diagram $[\alpha]^*$. Let VP be an element of S_n such that P is an element of type $(a_1, a_2, \ldots, a_s)$ and of weight b $(b = \sum a_i)$ and V is any permutation on the fixed symbols of P, and let W be an element of S_b with a_1-cycle, a_2-cycle, $\ldots$, a_s-cycle. We have by (6)

$$\chi_\alpha(VP) = h(\alpha, \alpha_0)\,\chi_{\alpha_0}(V). \tag{27}$$

Since $h(\alpha, \alpha_0)$ is determined by $(a_1, a_2, \ldots, a_s)$, we may set $h(\alpha, \alpha_0) = u(W)$. We then have by Thrall and Robinson **(18**; **14**; also cf. **6)**

$$u(W) = \sigma_\alpha\,\chi_{\alpha^*}(W), \tag{28}$$

where $\sigma_\alpha = \pm 1$ is the product of the parities of the b hooks of length p of $[\alpha]$. On the other hand we can prove that

$$\chi_{\alpha^*}(W) = \zeta_{\alpha^*}(W^*), \qquad W^* \in S_b^*. \tag{29}$$

Thus we may denote without confusion by $\chi_{\alpha^*}(G^*)$, $G^* \in S(b, p)$, the character of the ordinary irreducible representation of $S(b, p)$ corresponding to $[\alpha]^*$.

Let W_i ($i = 0, 1, 2, \ldots, k(b) - 1$) be a complete system of representatives for conjugate classes of S_b. If we denote by $n^*(W_i^*)$ the order of the normalizer $N^*(W_i^*)$ of W_i^* in $S(b, p)$ then it follows from (19) and (23) that

$$\sum_{\alpha^*} \chi_{\alpha^*}(W_i^*)\, \chi_{\alpha^*}(W_j^*) = \delta_{ij} n^*(W_i^*).$$

Evidently these relations are the orthogonality relations for the characters of $S(b, p)$.

4. Let V be any p-regular element of S_n and let W^* be any element of S_b^*. We have by (20)

$$\sum_{\alpha} \chi_{\alpha^*}(W^*)\, \chi_\alpha(V) = 0, \qquad [\alpha] \subset B. \tag{30}$$

It was shown in **(2)** that $S(b, p)$ possesses only one p-block. If we denote by

$$D^* = (d^*_{\alpha\lambda})$$

the decomposition matrix of $S(b, p)$, then (30) yields:

$$\sum_{\alpha} \sigma_\alpha d_{\alpha x}\, \chi_{\alpha^*}(W^*) = 0, \qquad [\alpha] \subset B, \tag{31}$$

$$\sum_{\alpha} \sigma_\alpha d^*_{\alpha\lambda}\, \chi_\alpha(V) = 0, \qquad [\alpha] \subset B, \tag{32}$$

and hence

$$\sum_{\alpha} \sigma_\alpha d_{\alpha x} d^*_{\alpha\lambda} = 0, \qquad [\alpha] \subset B. \tag{33}$$

Moreover we have the following

THEOREM 3. *Let B be a p-block of weight b and let $G = VP$ be an element of S_n such that P is any element of weight a different from b and V is any p-regular permutation on the fixed symbols of P. Then for any element $W^* \in S_b^*$,*

$$\sum_{\alpha} \sigma_\alpha\, \chi_\alpha(G)\, \chi_{\alpha^*}(W^*) = 0, \qquad [\alpha] \subset B.$$

This follows immediately from (19).

We obtain the generalization of the Murnaghan-Nakayama recursion formula for the character χ_{α^*} of $S(b, p)$ and this yields

THEOREM 4. *Let B be a p-block of weight b and let S be any element of $S(b, p)$ associated with a star diagram $[\beta]^* = [\lambda_0]\cdot[\lambda_1]\cdot \ldots \cdot[\lambda_{p-1}]$ such that $[\lambda_0] \neq [0]$. Then*

$$\sum_{\alpha} \sigma_\alpha\, \chi_\alpha(V)\, \chi_{\alpha^*}(S) = 0 \qquad (V \text{ in } S_n,\ p\text{-regular}).$$

Let R be any element of $S(b, p)$ associated with a star diagram $[\beta]^*$ such that $[\lambda_0] = [0]$. The number of conjugate classes of $S(b, p)$ which contain the element R defined above is $l^*(b)$. We denote by $R_1, R_2, \ldots, R_{l^*(b)}$ the representatives for these classes.

THEOREM 5. *Let $D = (d_{\alpha\lambda})$ be the decomposition matrix of a p-block B of weight b. Then*

$$d_{\alpha\lambda} = \sigma_\alpha \sum_{\kappa=1}^{l^*(b)} v_{\kappa\lambda}\, \chi_{\alpha^*}(R_\kappa), \qquad \text{for } [\alpha] \subset B,$$

where the $v_{\kappa\lambda}$ are complex numbers and are independent of α.

COROLLARY. *Let $D = (d_{\alpha\lambda})$ and $D' = (d'_{\alpha'\lambda})$ with $[\alpha]^* = [\alpha']^*$ be the decomposition matrices of p-blocks B and B′ of same weight respectively. Then*

$$d'_{\alpha'\nu} = \sigma_\alpha\, \sigma_{\alpha'} \sum_{\lambda=1}^{l^*(b)} w_{\nu\lambda}\, d_{\alpha\lambda}, \qquad \text{for } [\alpha'] \subset B',$$

where the $w_{\nu\lambda}$ are rational integers and $|w_{\nu\lambda}| = \pm 1$.

Consequently we have

THEOREM 6. *Two matrices of Cartan invariants corresponding to the p-blocks of same weight have the same elementary divisors.*

Example. The following is the U-matrix for the 2-block B of S_6 with 2-core [0].

$$\begin{array}{l} [6] \\ [5, 1] \\ [4, 2] \\ [4, 1^2] \\ [3^2] \\ [2^3] \\ [3, 1^3] \\ [2^2, 1^2] \\ [2, 1^4] \\ [1^6] \end{array}
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 3 & 1 & 0 \\
2 & 1 & 1 & 0 & 1 & -2 & 0 & -2 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & -1 & -3 & -1 & 0 \\
1 & 0 & 1 & -1 & 0 & 1 & 1 & 3 & -1 & 0 \\
2 & 1 & 1 & 0 & -1 & -2 & 0 & 2 & 0 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & -3 & 1 & 0 \\
1 & 1 & 0 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & 1 & -1
\end{bmatrix}$$

The matrix occupying the first three columns of this U-matrix is the decomposition matrix of B and the matrix occupying the last three columns is the matrix $(\sigma_\alpha\, \chi_{\alpha^*}\,(W_i^*))$ of $S(3, 2)$. We set

$$Q_1 = (1\ 2), \quad Q_2 = (3\ 4), \quad Q_3 = (5\ 6), \quad P = (1\ 2)(3\ 4)(5\ 6).$$

Then

$$W_0^* = 1, \quad W_1^* = (1\ 3)(2\ 4), \quad W_2^* = (1\ 3\ 5)(2\ 4\ 6),$$

$$Q_1, \quad W_1^*Q_3, \quad Q_1Q_2, \quad W_1^*Q_1, \quad P, \quad W_1^*Q_1Q_3, \quad W_2^*Q_1$$

form a complete system of representatives for conjugate classes of $S(3, 2)$. We then obtain easily Table I, showing the group characters χ_{α^*} of $S(3, 2)$ (cf. 5, p. 275).

TABLE I

class	$[1^3]\cdot[0]$	$[2,1]\cdot[0]$	$[3]\cdot[0]$	$[1^2]\cdot[1]$	$[2]\cdot[1]$	$[1]\cdot[1^2]$	$[1]\cdot[2]$	$[0]\cdot[1^3]$	$[0]\cdot[2,1]$	$[0]\cdot[3]$
element	1	(13)(24)	(135)(246)	(12)	(13)(24)(56)	(12)(34)	(1324)	(12)(34)(56)	(1324)(56)	(135246)
order	1	6	8	3	6	3	6	1	6	8
$[3]\cdot[0]$	1	1	1	1	1	1	1	1	1	1
$[0]\cdot[3]$	1	1	1	−1	−1	1	−1	−1	1	−1
$[2]\cdot[1]$	3	1	0	1	−1	−1	1	−3	−1	0
$[2,1]\cdot[0]$	2	0	−1	2	0	2	0	2	0	−1
$[1]\cdot[2]$	3	1	0	−1	1	−1	−1	3	−1	0
$[1^2]\cdot[1]$	3	−1	0	1	1	−1	−1	−3	1	0
$[0]\cdot[2,1]$	2	0	−1	−2	0	2	0	−2	0	1
$[1]\cdot[1^2]$	3	−1	0	−1	−1	−1	1	3	1	0
$[1^3]\cdot[0]$	1	−1	1	1	−1	1	−1	1	−1	1
$[0]\cdot[1^3]$	1	−1	1	−1	1	1	1	−1	−1	−1

The decomposition matrix D^* and the matrix C^* of Cartan invariants of $S(3, 2)$ are given by

$$D^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad C^* = \begin{bmatrix} 8 & 4 \\ 4 & 6 \end{bmatrix}.$$

The following are the D-matrices $(d_{\alpha\lambda})$ and $(d'_{\alpha'\lambda})$ for the 2-block of S_6 with 2-core [0] and the 2-block of S_7 with 2-core [1] respectively:

$$\begin{array}{l} [6] \\ [5, 1] \\ [4, 2] \\ [4, 1^2] \\ [3^2] \\ [2^3] \\ [3, 1^3] \\ [2^2, 1^2] \\ [2, 1^4] \\ [1^6] \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} [7] \\ [4, 2, 1] \\ [5, 1^2] \\ [5, 2] \\ [3^2, 1] \\ [3, 2^2] \\ [2^2, 1^3] \\ [3, 1^4] \\ [3, 2, 1^2] \\ [1^7] \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

We see from the table of the group characters χ_{α^*} of $S(3, 2)$ that

$$(d_{\alpha\lambda}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -3 & -1 & 0 \\ -2 & 0 & 1 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \\ -3 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & -\frac{1}{2} & 0 \\ \frac{4}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

There exists the following relation between $(d'_{\alpha'\lambda})$ and $(\sigma_\alpha \sigma_{\alpha'} d_{\alpha\lambda})$:

$$(d'_{\alpha'\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \\ -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ -2 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

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A SHORT PROOF OF THE CARTWRIGHT-LITTLEWOOD FIXED POINT THEOREM

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The purpose of this paper is to give a short proof of the Cartwright-Littlewood fixed point theorem (**2**, p. 3, Theorem A).

THEOREM A. *If T is a* (1-1) *continuous and orientation preserving transformation of the Euclidean plane E onto itself which leaves a bounded continuum M invariant and if M does not separate E, then some point of M is left fixed by T.*

We shall first prove a lemma suggested by Newman and proved by him independently (in an unpublished paper). We make use of his notation and some of his methods.

LEMMA 1. *If T is a* (1-1) *continuous and orientation preserving transformation of the Euclidean plane E onto itself which leaves a bounded continuum M invariant but leaves no point of M fixed and if M does not separate E, then there is a* (1-1) *continuous and orientation preserving transformation T' of E onto itself which coincides with T on M and leaves no point of E fixed.*

Proof. Since T, by hypothesis, leaves fixed no point of M, there exists a simple closed curve C_1 with inner domain D_1 containing M, such that if $x \epsilon \bar{D}_1$ then $T(x) \neq x$. Let C_2 and D_2 designate $T(C_1)$ and $T(D_1)$ respectively. By the Brouwer fixed point theorem for the 2-cell, neither of the domains D_1 and D_2 can contain the other. Hence $C_1 \cap C_2$ contains at least two points and, by a known theorem (**3**, p. 87; **4**, p. 168) the component G of $D_1 \cap D_2$ containing M has for its boundary a simple closed curve J. (See Fig. 1.) We may suppose J is the unit circle since it can be made so by a suitable topological mapping of the entire plane E.

For $r = 1, 2$ the components D_{ri} of $D_r - \bar{G}$ have each as frontier a simple closed curve composed of an arc L_{ri} of J and an arc of C_r with common endpoints. For each pair of subscripts r and i, let L'_{ri} be a circular arc of radius $1 - \delta$ with the same endpoints as L_{ri}, where $\delta > 0$ is small enough to ensure that no two arcs L'_{ri} meet except in endpoints. This is possible since the arcs L_{ri} of J are disjoint except for endpoints.

Let Δ_{ri} be the inner domain of $L_{ri} \cup L'_{ri}$. By a standard theorem there is a topological map ϕ_{ri} which maps $\bar{D}_{ri}$ onto $\bar{\Delta}_{ri}$ and leaves fixed each point of L_{ri}. Hence if

$$\bar{\Delta}_r = \bar{G} \cup \textstyle\bigcup_i \bar{\Delta}_{ri} \qquad (r = 1, 2)$$

the functions ϕ_r defined by

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$$\phi_r \mid \bar{G} = 1 \text{ (the identity map)}$$
$$\phi_r \mid \bar{D}_{rt} = \phi_{rt} \qquad (r = 1, 2)$$

are topological maps of $\bar{D}_r$ onto $\bar{\Delta}_r$ for $r = 1, 2$.

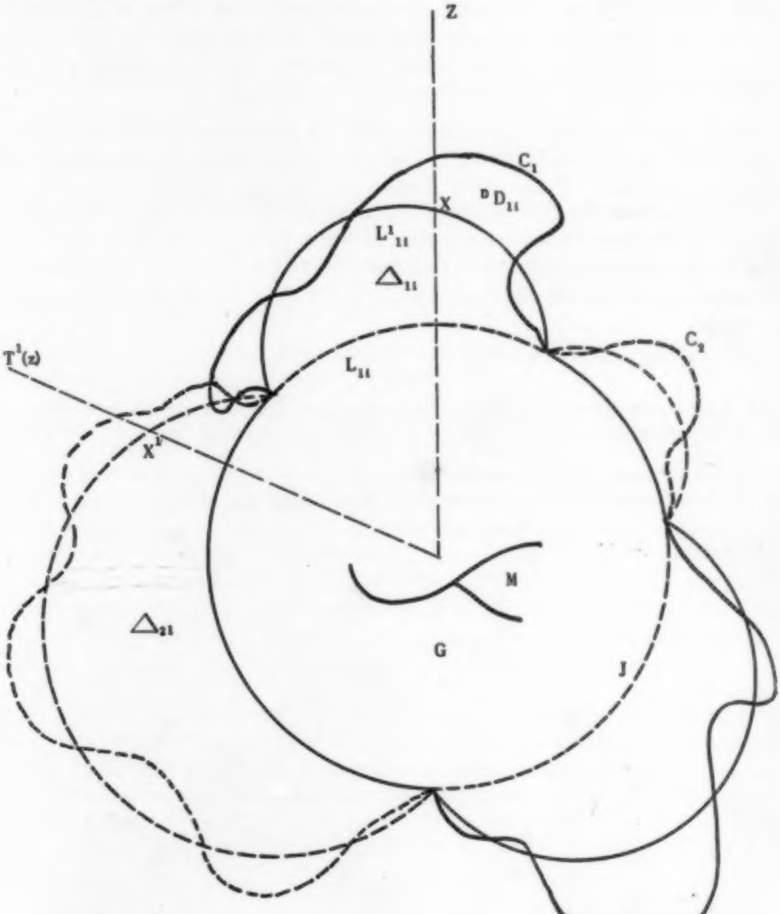

Figure 1

Let $T' : \bar{\Delta}_1 \to \bar{\Delta}_2$ be defined as $T' = \phi_2 \circ T \circ \phi_1^{-1}$. Then $T'|M = T|M$ since $T = T'$ in G. T' has no fixed point in $\bar{\Delta}_1$. For if $x \in \bar{G}$, $T'(x) = T(x) \neq x$; and if $x \in \bar{\Delta}_1 - \bar{G}$, $x \notin \bar{\Delta}_2 = T'(\bar{\Delta}_1)$.

Let T' be extended to the whole of E as follows: Let z be a point of $E - \bar{\Delta}_1$. Then z is expressible uniquely as $x + \rho\mu_z$, where $x \in \mathfrak{F} D_1$ and μ_z is the unit vector in the direction Ox, and $\rho > 0$. Let x' designate $T'(x)$ and define $T'(z) = x' + \rho\mu_z'$. This a topological mapping of E onto E. Suppose T' has a fixed point $z = T'(z)$. Then the directions from O to $z = x + \rho\mu_z$ and to $T'(z) = x' + \rho\mu_{z'}$ are the same and hence $\mu_z = \mu_{z'}$ and by subtraction $x = x' = T'(x)$ which contradicts the fact that T' has no fixed point in $\bar{\Delta}_1$. Hence $T'(z) \neq z$, and T' is the desired transformation.

Proof of Theorem A. Suppose that under the hypotheses of the theorem T leaves fixed no point of M. Then by Lemma 1 there is an orientation preserving homeomorphism T' of the plane E onto itself which coincides with T on M and leaves no point of E fixed. If p is a point of M then by a theorem of Brouwer (**1**, p. 45, Theorem 8) the set of points in the sequence $T'^n(p)$ $(n = 1, 2, \ldots)$ has no convergent subsequence. This contradicts the fact that M is compact. It follows that the assumption that T leaves no point of M fixed is false.

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ON LATTICE EMBEDDINGS FOR PARTIALLY ORDERED SETS

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1. Introduction. Let P be a set partially ordered by a (reflexive, antisymmetric, and transitive) binary relation $\prec$. Let $\mathfrak{K}$ be the family of all subsets K of P having the property that $x \in P$ and $y \in K$ and $y \prec x$ imply $x \in K$. Our principal object is to prove and apply the following:

THEOREM. *With respect to the partial ordering of $\mathfrak{K}$ by inclusion ($K_1 \prec K_2$ means $K_1 \supset K_2$),*

(1) *P is isomorphically embedded in $\mathfrak{K}$ preserving all suprema that exist in P, and*

(2) *$\mathfrak{K}$ is a complete distributive lattice.*

COROLLARY. *Every partially ordered set can be embedded in a complete distributive lattice, preserving suprema.*

This corollary is also a consequence of a two-stage embedding construction of MacNeille's (**2**, §11, 12) consisting of an initial completion by cuts, preserving both suprema and infima, followed by a certain complete-distributive-lattice embedding which preserves suprema and distributive infima. Our construction is much simpler than MacNeille's but does not in general preserve infima, even when they are distributive.

Following some related remarks concerning lattices of topologies in §3, an application of this theorem is indicated in §4. The author is indebted to the referee for suggestions leading to the recasting of results in essentially their present form, and to E. E. Floyd for a simplifying observation.

2. Proof of the theorem. We see at once that every subfamily $\mathfrak{K}_1$ of $\mathfrak{K}$ has an infimum (supremum) in $\mathfrak{K}$, namely the union (intersection) of the sets of the family $\mathfrak{K}_1$. That is, $\mathfrak{K}$ is a complete lattice. And now, since $\mathfrak{K}$ is a sublattice of the Boolean algebra of all subsets of P, it is obvious that $\mathfrak{K}$ is distributive.

It is easily seen that the correspondence

$$K(x) = \{y : x \prec y\} \qquad (x \in P)$$

is an isomorphism of P into $\mathfrak{K}$. We verify that it preserves suprema. Take any family $\{x_\alpha\}$ of elements of P having a supremum

$$x = \bigvee_\alpha x_\alpha$$

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in P. For each α, $x_\alpha < x$, so that

$$\{y: x < y\} \subset \bigcap_\alpha \{y: x_\alpha < y\}.$$

That is,

$$K(\bigvee_\alpha x_\alpha) \subset \bigvee_\alpha K(x_\alpha). \tag{2.1}$$

Now take any

$$z \in \bigvee_\alpha K(x_\alpha) = \bigcap_\alpha \{y: x_\alpha < y\}.$$

For each α, $x_\alpha < z$, whence

$$x = \bigvee_\alpha x_\alpha < z.$$

Thus

$$z \in K(\bigvee_\alpha x_\alpha).$$

That is,

$$K(\bigvee_\alpha x_\alpha) \supset \bigvee_\alpha K(x_\alpha),$$

which with (2.1) yields

$$K(\bigvee_\alpha x_\alpha) = \bigvee_\alpha K(x_\alpha)$$

as desired, completing the proof.

To see that this embedding does not always preserve distributive infima, let P be the rationals of the closed unit interval $[0, 1]$, partially ordered by $\leq$. The family $\{x_n\}$ of positive such rationals has the distributive infimum

$$0 = \bigwedge_n x_n,$$

so that

$$K(\bigwedge_n x_n) = \{y: 0 \leq y\} = [0, 1].$$

On the other hand,

$$\bigwedge_n K(x_n) = \bigcup_n \{y: x_n \leq y\} = (0, 1].$$

3. Lattices of topologies. We review some well-known facts. A topology T on a set S may be specified in any of several equivalent ways: in particular, by a closure function $C(X)$ on 2^S to 2^S such that

$$C(\phi) = \phi \qquad (\phi = \text{empty set}), \tag{3.1}$$

$$C(X) \cup C(Y) = C(X \cup Y), \tag{3.2}$$

$$X \subset C(X), \tag{3.3}$$

$$C(C(X)) = C(X). \tag{3.4}$$

The various topologies on S form a lattice $L_T(S)$ under the partial order $T_1 < T_2$ defined by the requirement

$$C_1(X) \supset C_2(X), \qquad X \in S. \tag{3.5}$$

This lattice is not in general distributive (**4**, p. 134). The definitive statement in this connection is very simple but seems not to have been elsewhere recorded:

(3.6) *Given a set S these statements are equivalent:*

(α) *$L_T(S)$ is modular.*

(β) *$L_T(S)$ is distributive.*

(γ) *The cardinality of S is <3.*

Proof. If (γ) holds, $L_T(S)$ has at most four elements and so is distributive by (**1**, p. 134, Theorem 2). That (β) implies (α) is trivial. To see that whenever (γ) fails (α) fails, assume $|S| > 3$, fix distinct points x and y of S, and consider these three closure topologies on S:

$$T_1:\ C_1(X) = X \text{ if } x \notin X;\ C_1(X) = X \cup \{y\}, \qquad x \in X,$$
$$T_2:\ C_2(X) = X \cup \{x\}, \qquad (X \neq \phi),$$
$$T_3:\ C_3(X) = X \cup \{y\}, \qquad (X \neq \phi).$$

One verifies easily that

$$(T_1 \vee T_2) \wedge T_3 = T_3 < T_1 = T_1 \vee (T_2 \wedge T_3).$$

Since $|S| > 3$ implies $T_1 \neq T_3$, this contradicts modularity.

By dropping requirement (3.4) on closure functions, Wada (**5**) arrived at the larger lattice $L_A(S)$ of what he termed the "additive topologies" on S; and by dropping (3.3) as well he obtained the still larger lattice $L(S)$ of Tukey topologies (**3**, p. 24). He observed that $L(S)$ is complete and distributive and that it embeds $L_A(S)$ as a sublattice and $L_T(S)$ as a partially ordered set, preserving suprema.

4. Channel structures. Application of our embedding theorem to $L_A(S)$ yields a suprema-preserving embedding of $L_A(S)$ in a complete distributive lattice $L_C(S)$ whose elements, termed *channel structures* on S, are of considerable intrinsic interest. The notion of channel structure is due, in its original somewhat different form, to McShane[1]. Here we content ourselves with a very brief indication of this original form.

We first note (cf. **3**, p. 19, Theorem 3.14) that an additive topology on a set S can be equivalently defined by a neighbourhood function $\mathfrak{N}$ associating with each point $x \in S$ a non-empty class $\mathfrak{N}(x)$ of subsets of S such that

(4.1) $x \in N$ for each $N \in \mathfrak{N}(x)$;

(4.2) if $S \supset M \supset N$ and $N \in \mathfrak{N}(x)$, then $M \in \mathfrak{N}(x)$; if M, $N \in \mathfrak{N}(x)$, then $M \cap N \in \mathfrak{N}(x)$.

[1]Channel structures will form the subject of a forthcoming joint study by E. J. McShane, E. E. Floyd, and the present author.

The partial order (3.5) on $L_A(S)$ is then equivalently defined (**3**, p. 24) by the requirement

$$\text{for each } y \in S,\ \mathfrak{N}_1(x) \subset \mathfrak{N}_2(x). \tag{4.3}$$

Suppose now we define a *channel to* $x \in S$ as a non-empty class $\mathfrak{N}(x)$ of subsets of S satisfying (4.1) and (4.2). It is then not difficult to see that each channel structure (= element of $L_C(S)$) on S consists essentially of a function $\mathcal{N}$ which assigns to each $x \in S$ a collection $\mathcal{N}(x)$ of channels to x with the following property: if $\mathfrak{N}_1(x)$ and $\mathfrak{N}_2(x)$ are channels to x and $\mathfrak{N}_1(x) \in \mathcal{N}(x)$ and $\mathfrak{N}_2(x) \supset \mathfrak{N}_1(x)$, then $\mathfrak{N}_2(x) \in \mathcal{N}(x)$. We conclude by remarking that because the lattice $L_A(S)$ has a unit I (the discrete topology on S), none of these collections $\mathcal{N}(x)$ can be empty.

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DIFFERENTIAL EQUATIONS OF NON-INTEGER ORDER

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Introduction. In §1, we define a differential-integral operator, which for positive real indices is commonly known as the Liouville-Riemann generalized integral. For positive integer indices, we obtain an iterated integral. For negative real indices we obtain the Riemann-Holmgren **(5; 9)** generalized derivative, which for negative integer indices gives the ordinary derivative of order corresponding to the negative of such an integer. Following M. Riesz **(10)** we extend these ideas to include complex indices. An equation involving this operator where the real part of the index is negative will be called a differential equation of non-integer order. It is to be noted that the distinction between a differential equation and an integral equation disappears when the index is not an integer, although for rational indices such an equation may be transformed into an ordinary differential equation. The Riemann-Holmgren form of the definition itself involves both differentiation and integration of the ordinary kind and contributes to the breaking down of this distinction.

A classical example of a differential equation of non-integer order is the inverse of the Abel integral equation **(1**, p. 8**)**; that is, consider the solution as the "differential" equation, then the integral equation becomes the solution of this equation. An example of such an equation was discussed by Post **(8)** and Davis **(3)** related such equations to Volterra integral equations. In this paper we are concerned with equations of irrational and even complex order. For the fundamental equation (A) of §2 we note that the only solution which is continuous at the "lower limit" a of our operator is the trivial one and we find that it is of interest to allow a singularity at a. In §2 we show that for the index α real and between 0 and 1 the solutions of (A) have many of the same properties as e^{-z}, which is the principal solution for $\alpha = 1$. In §4 we use properties established in §2 to add to the discussions by Mittag-Leffler **(7)** and Wiman **(12)** on the behavior of the complex entire function $E_\alpha(z)$ for $0 < \alpha < 1$ on the real negative z-axis. Then for $1 < \alpha$ we apply theorems of Mittag-Leffler and Wiman to establish the behavior of our solutions for this range of the index α.

The operator with non-integer positive real indices makes its appearance in solutions of partial differential equations, for example, the Euler-Poisson equation **(2**, p. 54**)** which plays an important role in the theory of partial differential equations of mixed types as developed by Tricomi **(11)**. This

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operator is the one-dimensional case of the n-dimensional operator of M. Riesz **(10)**. Because of this and since Holmgren **(5)** published the idea of generalized differentiation before Riemann's work **(9**, pp. 331-344**)** appeared in print we call this operator the one-dimensional Holmgren-Riesz Transform.

1. Properties of the Holmgren-Riesz Transform. Let a and b be real numbers, $a < b$; $L(a, b)$ be the class of all complex functions of a real variable x which are summable (Lebesgue) on $a < x < b$; $\alpha = \alpha_1 + i\alpha_2$ be a complex number; the real part $R\alpha = \alpha_1$; for A real and positive, $A^{\alpha} = A^{R\alpha}(\cos\alpha_2 \ln A + i \sin \alpha_2 \ln A)$ and $||\alpha|| = \max(|\alpha_1|, |\alpha_2|)$, i.e. $||\alpha||$ is not less than the two non-negative numbers, $|\alpha_1|, |\alpha_2|$.

If $f(t)$ is a function which is defined a.e. on $a < t < b$ then the one-dimensional Holmgren-Riesz Transform of index α will be represented by the notation: $I(\alpha; a, b \,|\, f)$.

Definition 1.1. If $0 < R\alpha$, then

$$I(\alpha;\, a, b|f) = \int_a^b f(t)\, \frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}\, dt,$$

provided that this integral (Lebesgue) exists.

An extension of Definition 1.1 is:

Definition 1.2. If $R\alpha \leqslant 0$; n is the smallest positive integer $> -R\alpha$; then

$$I(\alpha;\, a, b|f) = D_x^n\, I(n + \alpha;\, a, x|f)$$

at $x = b$, provided that $I(n + \alpha; a, x \,|\, f)$ and its first $(n - 1)$ derivatives exist in a segment, $|b - x| < h$, and the nth derivative exists at $x = b$.

Example 1.1. For complex β, $R\beta > -1$ and $x > a$:

$$I\left(\alpha;\, a, x \,\middle|\, \frac{(t-a)^{\beta}}{\Gamma(\beta+1)}\right) = \begin{cases} \dfrac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, & R(\alpha+\beta) \neq \text{negative integer}, \\[2ex] 0 \qquad , & R(\alpha+\beta) = \text{negative integer}. \end{cases}$$

The following is an extension of a theorem due to Hardy **(4)** which was stated only for real numbers α and β.

THEOREM 1.1. *If* $0 < R\alpha$, $R\alpha \leqslant R\beta$; $f(x)$ *belongs to* $L(a, b)$; *and* $I(R\alpha; a, b \,|\, ||f||)$ *exists then* $I(\beta; a, b \,|\, f)$ *exists.*

Proof. Let $F(x) = \max\,[||f(x)||,\, ||f(x)||\,(b - x)^{R\alpha-1}]$, then $F(x)$ is in $L(a, b)$. Also, $f(x)\,(b - x)^{\beta-1}$ is measurable on $a < x < b$ and the absolute value of each component is not greater than $F(x)$. Hence, $f(x)\,(b - x)^{\beta-1}$ is in $L(a, b)$.

THEOREM 1.2. *Let* $0 < R\alpha$; *and* β *real;* $0 \leqslant \beta \leqslant R\alpha$; $0 < M$; $a < x_0 < b$ *and* $f(x)$ *belong to* $L(a, b)$. *Then*

(a) *If* $a < d < x_0$ *and* $\|f(x)\| \leq M(x_0 - x)^{-\beta}$ *on* $d \leq x \leq x_0$ *then* $I(\alpha; a, x \mid f)$ *exists on* $d < x \leq x_0$ *and has left-hand continuity at* $x = x_0$;

(b) *If* $x_0 < d \leq b$; $\|f(x)\| \leq M(x - x_0)^{-\beta}$ *for* $x_0 < x < d$ *and* $I(R\alpha; a, x_0 \mid f)$ *exists, then* $I(\alpha; a, x \mid f)$ *exists on* $x_0 \leq x \leq d$ *and has right-hand continuity at* $x = x_0$.

Proof. Part (a). Note that $I(\alpha; a, x \mid f)$ exists on $d \leq x \leq x_0$ and

$$I(\alpha;\, a, x_0|f) - I(\alpha;\, a, x|f)$$
$$= \int_a^x f(t)\, dt \int_x^{x_0} \frac{(s - t)^{\alpha-2}}{\Gamma(\alpha - 1)} ds + \int_x^{x_0} f(t)\, \frac{(x_0 - t)^{\alpha-1}}{\Gamma(\alpha)}\, dt$$

on $d \leq x < x_0$. Write the first integral:

$$\int_a^x = \int_a^d + \int_d^x$$

and it is clear that each of the three integrals on the right-hand side approaches zero as x approaches x_0. Part (b) is proved in a similar manner with the additional observation that $I(\alpha; a, x_0 \mid f)$ exists since $I(R\alpha; a, x_0 \mid f)$ exists. That this does not follow from the other hypotheses of part (b) is illustrated by the following:

Example 1.2. Let $0 < \alpha < 1$ and $f(x) = (1 - x)^{-\alpha}$ for $0 \leq x < 1$, $f(x) = 0$ for $1 \leq x$. Then $f(x)$ belongs to $L(0, 1)$ and $I(\alpha; 0, 1 \mid f)$ does not exist. Furthermore, for $1 < x$,

$$I(\alpha;\, 0, x|f) > -\frac{\ln(x - 1)}{\Gamma(\alpha)}$$

which increases without bound as x approaches 1.

From Theorem 1.2 we have, immediately:

COROLLARY 1.2.1. *If* $R\alpha > 0$ *and* $f(x)$ *is continuous on* $a \leq x \leq b$ then $I(\alpha; a, x \mid f)$ *exists and is continuous with respect to* x *on* $a \leq x \leq b$.

The next property is an extension of another theorem due to Hardy **(4)** which was stated for real summable $f(x)$, real α and $\beta = 0$. Riesz **(10)** has discussed this theorem and its corollaries for continuous $f(x)$ and complex α and β.

THEOREM 1.3. *If* $R\alpha > 0$; $R\beta \geq 0$ *and* $f(x)$ *belongs to* $L(a, b)$ *then:*

(a) $I(\alpha; a, x \mid f)$ *exists* $\begin{cases} \textit{everywhere on } a \leq x \leq b, \textit{ for } R\alpha \geq 1 \\ \quad\textit{a.e. on } a \leq x \leq b, \textit{ for } R\alpha < 1 \end{cases}$

(b) $I(\beta + 1; a, b \mid I(\alpha; a, t \mid f)) = I(\alpha + \beta + 1; a, b \mid f)$.

Proof. If $R\alpha \geq 1$, then by Theorem 1.1, $I(\alpha; a, x \mid f)$ exists everywhere on $a \leq x \leq b$. For the general case, $R\alpha > 0$, we will follow the argument suggested by Hardy **(4**, p. 146**)** for his restricted case. Since $f(x)$ is in $L(a, b)$ then so is

$||f(x)||$. Let $g_n(x) = \min [||f||, n]$, $a < x < b$; $K_n(x) = \min [x^{R\alpha-1}, n]$, $0 < x < b - a$; $K_n(0) = n$. Then $g_n(t)K_n(x - t)(b - x)^{R\beta}$ is a summable function of x and t over the triangle T: $a < x < b$, $a < t < x$; since it is the product of bounded summable functions. Then, by Fubini's Theorem **(6)** we have that:

$$\begin{aligned}\int_T g_n(t)\, K_n(x - t)(b - x)^{R\beta} dT &= \int_a^b dx \int_a^x g_n(t)\, K_n(x - t)(b - x)^{R\beta} dt \\ &= \int_a^b dt \int_t^b g_n(t)\, K_n(x - t)(b - x)^{R\beta} dx \\ &< \int_a^b g_n(t)\, dt \int_t^b (x - t)^{R\alpha-1}(b - x)^{R\beta} dx \\ &= B(R\alpha, R\beta + 1) \int_a^b ||f(t)||(b - t)^{R\alpha+R\beta} dt.\end{aligned}$$

Since $g_n(t)\, K_n(x - t)(b - x)^{R\beta}$ is a non-decreasing sequence of summable functions over T, then

$$\int_T ||f(t)||(x - t)^{R\alpha-1}(b - x)^{R\beta} dx dt$$

exists. Then since $f(t)\, (x - t)^{\alpha-1}\, (b - x)^{\beta}$ is a measurable function of x and t over T, each of whose components is bounded by $||f(t)||\, (x - t)^{R\alpha-1}\, (b - x)^{R\beta}$, we see that each of the following exist and

$$\begin{aligned}\int_T f(t)(x - t)^{\alpha-1}(b - x)^{\beta} dx dt &= \int_a^b dx \int_a^x f(t)(x - t)^{\alpha-1}(b - x)^{\beta} dt \\ &= \int_a^b f(t) dt \int_t^b (x - t)^{\alpha-1}(b - x)^{\beta} dx,\end{aligned}$$

from which (b) follows easily.

Corollary 1.3.1. *If* $R\alpha > 0$, $R\beta > 0$ *and* $f(x)$ *belongs to* $L(a, b)$ *then on* $a < x < b$:

(a) $\displaystyle\int_a^x I(\alpha;\, a, t|f)\, dt = I(\alpha + 1;\, a, x|f)$;

(b) *If* $R\alpha > 1$ or $\alpha = 1$; $I(\alpha; a, x\,|\,f)$ *is absolutely continuous in* x;

(c) $D_x I(\alpha + 1; a, x\,|\,f) = I(\alpha; a, x\,|\,f) \begin{cases} \textit{everywhere, if } R\alpha > 1, \quad \alpha = 1, \\ \textit{a.e.} \quad \textit{if } R\alpha < 1, \quad \alpha \neq 1; \end{cases}$

(d) $I(\alpha + \beta; a, x\,|\,f) = I(\beta; a, x\,|\, I(\alpha; a, t\,|\,f)) \begin{cases} \textit{everywhere, if } R(\alpha + \beta) > 1, \\ \qquad\qquad \alpha + \beta = 1, \\ \textit{a.e.} \quad \textit{if } R(\alpha + \beta) < 1, \\ \qquad\qquad \alpha + \beta \neq 1. \end{cases}$

Theorem 1.4. *If* $f(x)$ *is absolutely continuous on* $a < x < b$ *and* $R\alpha > 0$ *then*

$$I(\alpha;\, a, x|f) = \frac{f(a)(x - a)^{\alpha}}{\Gamma(\alpha + 1)} + I(\alpha + 1, a, x|f').$$

Proof. Use integration by parts **(6)**.

COROLLARY 1.4.1. *If* $R\alpha > 0$; $f(x)$ *is* $L(a, b)$, *then*

$$I\left(\alpha;\ a, x \,\middle|\, \int_a^t f\right) = I(\alpha + 1;\ a, x|f).$$

COROLLARY 1.4.2. *If* $f(x)$ *is absolutely continuous on* $a < x < b$ *and* $R\alpha > 0$ *then* $I(\alpha; a, x \mid f)$ *is absolutely continuous on* $a < x < b$.

COROLLARY 1.4.3. *If* n *is a positive integer;* $f(x)$ *is of class* $C^{(n)}$ *on* $a < x < b$ *and belongs to* $L(a, b)$ *and* $R\alpha > 0$ *then* $I(\alpha; a, x \mid f)$ *is of class* $C^{(n)}$ *on* $a < x < b$ *and belongs to* $L(a, b)$.

Proof. Let $n = 1$. $I(\alpha; a, x \mid f)$ belongs to $L(a, b)$ from Theorem 1.3. Let $a < x_0 < b$ then

$$I(\alpha;\ a, x|f) = \int_a^{x_0} f(t)\, \frac{(x - t)^{\alpha-1}}{\Gamma(\alpha)}\, dt + \frac{f(x_0)(x - x_0)^{\alpha}}{\Gamma(\alpha + 1)} + I(\alpha + 1;\ x_0, x|f'); \qquad x > x_0.$$

Since $f'(x)$ is continuous on $x_0 < x < b$ then $I(\alpha; x_0, x \mid f')$ is continuous and $D_x I(\alpha; a, x \mid f)$ is continuous on $x_0 < x < b$. Since this is true for any such x_0, the conclusion follows for $n = 1$. By induction the theorem can be established for any positive integer n.

THEOREM 1.5. *If* $R\alpha > 0$ *and* $f(x)$ *is in* $L(a, b)$ *then*

$$I(-\alpha; a, x \mid I(\alpha; a, t \mid f)) = f(x), \text{ a.e. on } a < x < b.$$

Proof. Let n be the smallest integer $> R\alpha$, then applying Definition 1.2 and Corollary 1.3.1 we have:

$$D_x^n\, I(n - \alpha;\ a, x|I(\alpha;\ a, t|f)) = D_x^n\, I(n;\ a, x|f) = f(x), \text{ a.e. on } a < x < b.$$

THEOREM 1.6. *If* $R\alpha > 0$; n *is the smallest integer* $> R\alpha$; $f(x)$ *is in* $L(a, b)$ *and* $I(1 - \alpha; a, x \mid f)$ *exists and is absolutely continuous on* $a < x < b$, *then* $I(i - \alpha; a, a^+ \mid f) = K_i$ *exists for* $i = 1, 2 \ldots n$; $I(-\alpha; a, x \mid f)$ *exists a.e. on* $a < x < b$, *is in* $L(a, b)$ *and*

$$I(\alpha;\ a, x|I(-\alpha;\ a, t|f)) = f(x) - \sum_{p=1}^{n} \frac{K_p(x - a)^{\alpha-p}}{\Gamma(\alpha - p + 1)}, \text{ a.e. on } a < x < b.$$

Furthermore, the equality holds everywhere on $a < x < b$, *if, in addition,* $f(x)$ *is continuous on* $a < x < b$.

Proof. Let $g(x) = I(-\alpha; a, x \mid f)$ a.e. on $a < x < b$. Since $I(1-\alpha; a, x \mid f)$ is absolutely continuous on $a < x < b$ then $I(1 - \alpha; a, a^+ \mid f)$ exists and $I(1 - \alpha; a, x \mid f) = K_1 + I(1, a, x \mid g)$ on $a < x < b$. If $n > 1$, then by continuing this process we have

$$I(n - \alpha;\ a, x|f) = \sum_{p=1}^{n} \frac{K_p(x - a)^{n-p}}{\Gamma(n - p + 1)} + I(n;\ a, x|g) \text{ on } a < x < b.$$

Then

$$I(n;\, a, x|f) = I(\alpha;\, a, x|I(n-\alpha;\, a, t|f)) = \sum_{p=1}^{n} \frac{K_p(x-a)^{n-p+\alpha}}{\Gamma(n-p+\alpha+1)} + I(n+\alpha;\, a, x|g)$$

and

$$(x) = D_x^n\, I(n;\, a, x|f) = \sum_{p=1}^{n} \frac{K_p(x-a)^{\alpha-p}}{\Gamma(\alpha-p+1)} + I(\alpha;\, a, x|g) \quad \text{a.e. on } a < x < b.$$

We solve for $I(\alpha; a, x \mid g)$ and obtain the desired equality.

If $0 < \mathrm{R}\alpha < 1$ and $f'(x)$ exists on $a < x < b$ and is continuous at $x = a$ then, by Theorem 1.4 and Corollary 1.2.1, $K_1 = 0$ and

$$I(\alpha; a, x \mid I(-\alpha, a, t \mid f)) = f(x)$$

on $a < x < b$. For $K_1 \neq 0$, note the following:

Example 1.3. Let $0 < \mathrm{R}\alpha < 1$ and

$$f(x) = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}, \qquad x > a,$$

then $K_1 = 1$ and

$$I(\alpha;\, a, x|I(-\alpha;\, a, t|f)) = f(x) - K_1 \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} = 0, \qquad x > a.$$

THEOREM 1.7. *If* $\mathrm{R}\alpha > 0$; λ *is a complex number;* $f(x)$ *is in* $L(a, b)$; $a < x_0 < b$ *and* $I(\mathrm{R}\alpha; a, x_0 \mid f)$ *exists then*

(a) $\quad ||I(\alpha;\, a, x_0|f)|| < A \cdot I(\mathrm{R}\alpha;\, a, x_0|\, ||f||)$, *where* $A = \dfrac{\Gamma(\mathrm{R}\alpha)}{|\Gamma(\alpha)|}$;

(b) $\quad \sum_{p=0}^{\infty} \lambda^p\, I(p\alpha;\, a, x|f)$ *converges absolutely and uniformly a.e. on* $a < x < b$.

Proof. Since $f(t)\,(x_0 - t)^{\mathrm{R}\alpha-1}$ is in $L(a, x_0)$ then so are $f(t)\,(x_0 - t)^{\alpha-1}$ and $||f(t)||\,(x_0 - t)^{\mathrm{R}\alpha-1}$; hence $I(\alpha; a, x_0 \mid f)$ and $I(\mathrm{R}\alpha; a, x_0 \mid ||f||)$ exist. Furthermore, by considering separately the real and imaginary components the inequality (a) follows. For part (b) let m be the smallest positive integer such that $m\alpha > 1$. Then, $||I(m\alpha; a, x \mid f)||$ is continuous and, hence, bounded (say by M) on $a < x < b$ and for $n > m$ we have that

$$I(n\alpha; a, x \mid f) = I(\overline{n - m}\alpha; a, x \mid I(m\alpha; a, t \mid f))$$

and

$$||I(n\alpha;\, a, x|f)|| < A^{n-m} \cdot I(\mathrm{R}(n-m)\alpha;\, a, x|M) = \frac{MA^{n-m}(x-a)^{\mathrm{R}(n-m)\alpha}}{\Gamma(\mathrm{R}(n-m)\alpha+1)}.$$

The conclusions of part (b) follow easily with the use of the following inequality:

LEMMA. *If X and β are real, positive numbers, then* $\Gamma(\beta + 1) > X^\beta e^{-X}$.

Proof.

$$\Gamma(\beta+1) = \int_0^\infty e^{-x} x^\beta\, dx > \int_X^\infty e^{-x} x^\beta\, dx > X^\beta e^{-X}.$$

This list of properties of the Transform will be concluded with the following theorem which makes use of the discussion of modes of convergence by McShane (**6**, pp. 160-168).

THEOREM 1.8. *If* $R\alpha > 0$; $f_1(x), f_2(x), \ldots$ *is a sequence of functions in* $L(a, b)$ *which converges almost everywhere on* $a < x < b$ *to a function* $f(x)$ *in* $L(a, b)$ *and there exists a non-negative real function* $g(x)$ *in* $L(a, b)$ *such that* $\|f_n(x)\| < g(x)$ *for all* n *and all* x *on* $a < x < b$ *then*

$$\lim_{n\to\infty} I(\alpha;\, a, x|f_n) = I(\alpha;\, a, x|f)$$

almost everywhere on $a < x < b$ *and, hence, almost uniformly.*

Proof. Let x be a number such that $a < x < b$ and $I(R\alpha; a, x \mid g)$ exists. Let $h_n(t)$ be one of the (real or imaginary) components of $f_n(t)(x-t)^{\alpha-1}$ and $h(t)$ be the corresponding component of $f(t)(x-t)^{\alpha-1}$ on $a < t < x$. Then

$$|h_n(t)| < 2g(t)\,(x-t)^{R\alpha-1}$$

and since $h_n(t)$ is measurable and $I(R\alpha; a, x \mid g)$ exists then $\int_a^x h_n(t)\,dt$ exists and, similarly, $\int_a^x h(t)\,dt$ exists. Furthermore $h_n(t)$ converges to $h(t)$ almost everywhere on $a < t < x$ and, hence (**6**, p. 168),

$$\lim_{n\to\infty} \int_a^x h_n(t)\,dt = \int_a^x h(t)$$

and it follows that

$$\lim_{n\to\infty} I(\alpha;\, a, x|f_n) = I(a;\, a, x|f).$$

Since the above discussion applies to the interval $a < x < b$ except at most a subset of measure zero, the convergence holds almost everywhere. Finally, the transforms are all in $L(a, b)$ which ensures that the convergence is almost uniform on $a < x < b$ (**6**, p. 164).

2. Linear differential equations of non-integer order. We shall be concerned with the following linear integral-differential equation for $R\alpha > 0$, any complex number λ and $h(x)$ in $L(a, b)$:

(A) $$I(-\alpha; a, x|y) + \lambda y = h(x).$$

Because of Theorem 1.6 on inverse operations, we shall impose boundary conditions of the type:

(B) $I(i-\alpha; a, a^+|y) = K_i$; $\quad i = 1,2,\ldots,n$; where $n - 1 < R\alpha < n$.

Definition 2.1. A function $f(x)$ is said to be an *L-solution* of (A) provided that it belongs to $L(a, b)$; $I(1-\alpha; a, x|f)$ exists and is absolutely continuous on $a < x < b$ and equation (A) is satisfied by $y = f(x)$ a.e. on $a < x < b$. $f(x)$ is said to be a *unique* solution of (A) and (B) provided that any other solution $g(x)$ differs from $f(x)$ only on a null sub-set of $a < x < b$.

Definition 2.2. A function $f(x)$ is said to be an *R-solution* of (A) provided that it is an L-solution which satisfies (A) on $a < x < b$.

Suppose that $\lambda, K_1, K_2, \ldots K_n, \alpha$ are complex numbers; $h(x)$ is in $L(a, b)$; n is the positive integer such that $n - 1 < \mathrm{R}\alpha < n$; $\alpha \neq n - 1$ and $f(x)$ is an L-solution of (A) and (B). Then by Theorem 1.6:

$$f(x) = \sum_{p=1}^{n} \frac{K_p(x-a)^{\alpha-p}}{\Gamma(\alpha - p + 1)} + I(\alpha;\, a, x|h) - \lambda I(\alpha;\, a, x|f).$$

By successive substitutions it follows that for any positive integer m and a.e. on $a < x < b$:

$$f(x) = \sum_{q=1}^{m}\sum_{p=1}^{n} (-\lambda)^{q-1}\frac{K_p(x-a)^{q\alpha-p}}{\Gamma(q\alpha - p + 1)} + \sum_{q=1}^{m} (-\lambda)^{q-1} I(q\alpha;\, a, x|h) + (-\lambda)^m I(m\alpha;\, a, x|f).$$

Then, using Theorem 1.7 we have $f(x) = g(x)$ a.e. on $a < x < b$ where

$$g(x) = \sum_{p=1}^{n} K_p \sum_{q=1}^{\infty} (-\lambda)^{q-1}\frac{(x-a)^{q\alpha-p}}{\Gamma(q\alpha - p + 1)} + \sum_{q=1}^{\infty} (-\lambda)^{q-1} I(q\alpha;\, a, x|h), \quad \text{on } a < x < b.$$

This establishes that if there is an L-solution it must be equal to $g(x)$ a.e. on $a < x < b$. Therefore all that remains to be done in order to show that there is a unique L-solution is to show that $g(x)$ is an L-solution.

Each of these series converges uniformly and absolutely a.e. on $a < x < b$, for all values of λ, which allows the interchange of order of the operations which follow.

By use of Theorems 1.5, 1.8 and Example 1.1 we have: $g(x)$ is in $L(a, b)$ and

$$I(-\alpha;\, a, x|g) = \sum_{q=2}^{\infty}\sum_{p=1}^{n} (-\lambda)^{q-1}\frac{K_p(x-a)^{(q-1)\alpha-p}}{\Gamma((q-1)\alpha - p + 1)} + h(x) - \lambda \sum_{q=1}^{\infty} (-\lambda)^{q-1} I(q\alpha;\, a, x|h),$$

which reduces to $I(-\alpha; a, x|g) = h(x) - g(x)$, a.e. on $a < x < b$. Furthermore by computing $I(i - \alpha; a, x|g)$, we see that

$$I(i - \alpha; a, a^+|g) = K_i, \qquad i = 1, 2, \ldots, n.$$

Thus, $g(x)$ is a unique L-solution of (A) and (B).

Let

$$U_p(x;\, \lambda) = \sum_{q=1}^{\infty} \frac{(-\lambda)^{q-1} x^{q\alpha-p}}{\Gamma(q\alpha - p + 1)}$$

for $x > 0$. Then $U_p(x - a; \lambda)$ is an R-solution of (A) and (B) where $K_i = 1$ for $i = p$ and $K_i = 0$ for $i \neq p$. Furthermore

$$\sum_{q=1}^{\infty} (-\lambda)^{q-1} I(q\alpha;\, a, x|h) = \int_a^x h(t)\, U_1(x - t;\, \lambda)\, dt.$$

If $\alpha = n - 1$, an integer, the case is that of ordinary linear differential equations.

THEOREM 2.1. *If* $R\alpha > 0$; *n is the smallest positive integer* $> R\alpha$; λ *is a complex number;* $K_1, K_2, \ldots, K_n$ *is a complex number sequence; and* $h(x)$ *is in* $L(a, b)$ *then*

$$f(x) = \sum_{p=1}^{n} K_p\, U_p(x - a;\ \lambda) + \int_a^x h(t)\, U_1(x - t;\ \lambda)\, dt$$

is the unique L-solution of (A) and (B) on $a < x \leqslant b$.

COROLLARY 2.1.1. *If, in addition to the hypothesis of Theorem* 2.1, $h(x)$ *is continuous on* $a < x \leqslant b$ *then* $f(x)$ *is a unique R-solution of (A) and (B) on* $a < x \leqslant b$.

3. Behavior of solutions of the homogeneous equation where $0 < \alpha < 1$. Let α be a real number between 0 and 1 and let $Y(x)$ be the unique R-solution of (A) and (B) on $a < x \leqslant b$ where $K_1 = 1$, $\lambda = 1$, and $a = 0$ and $h(x) = 0$ for $x > 0$:

$$Y(x) = U_1(x;\ 1) = \sum_{q=1}^{\infty} \frac{(-1)^{q-1} x^{q\alpha-1}}{\Gamma(q\alpha)}, \qquad x > 0.$$

Since $I(1 - \alpha; 0, 0^+ \,|\, Y) = 1$; it is clear that for some $\bar{x} > 0$: $Y(x) > 0$ on $0 < x < \bar{x}$. Suppose that $Y(x)$ has a zero on $0 < x < b$ and let x_0 be the smallest such zero.

Then $Y(x) > 0$ for $0 < x < x_0$ and $Y(x_0) = 0$. Recall that

$$D_x I(1 - \alpha; 0, x \,|\, Y) + Y(x) = 0,$$

Then,

$$I(1 - \alpha; 0, x \,|\, Y) + I(1, 0, x \,|\, Y) = 1,$$

$$I(1 - \alpha;\ x_0, x_0^+ \,|\, Y) + I(1, 0, x \,|\, Y) = 1 - \int_0^{x_0} Y(t)\, \frac{(x - t)^{-\alpha}}{\Gamma(1 - \alpha)}\, dt, \qquad x > x_0,$$

and $I(1 - \alpha; x_0, x_0^+ \,|\, Y) = 0$, since $Y(x)$ is continuous at $x = x_0$. Let $0 < d < x_0$, then since $Y(x_0) = 0$ we have the integration by parts:

$$h(x) = -\int_0^d \frac{Y(t)(x - t)^{-\alpha-1}}{\Gamma(-\alpha)}\, dt - \frac{Y(d)(x - d)^{-\alpha}}{\Gamma(1 - \alpha)} - \int_d^{x_0} \frac{Y'(t)(x - t)^{-\alpha}}{\Gamma(1 - \alpha)}\, dt, \qquad x > x_0,$$

and, hence, $h(x)$ is continuous on $x_0 < x \leqslant b$ and is in $L(x_0, b)$.

Now, from Corollary 2.1.1 it follows that

$$Y(x) = I(1 - \alpha;\ x_0, x_0^+ \,|\, Y) \cdot Y(x - x_0) + \int_{x_0}^x h(t)\, Y(x - t)\, dt, \quad x_0 < x \leqslant b.$$

Note that $h(x) > 0$ for $x > x_0$ and $Y(x - t) > 0$ for $x_0 < t < x < 2x_0$. Therefore $Y(x) > 0$ for $x_0 < x < 2x_0$ and $Y'(x_0) = 0$. However,

$$Y(x_0)\, \frac{(x - x_0)^{1-\alpha}}{\Gamma(2 - \alpha)} + I(2 - \alpha;\ x_0, x \,|\, Y') + I(1;\ 0, x \,|\, Y)$$

$$= 1 - \int_0^{x_0} Y(t)\, \frac{(x - t)^{-\alpha}}{\Gamma(1 - \alpha)}\, dt, \qquad x > x_0;$$

$$I(-\alpha;\ x_0, x \,|\, Y') + Y'(x) = -\int_0^{x_0} \frac{Y(t)(x - t)^{-2-\alpha}}{\Gamma(-\alpha-1)}\, dt = h_1(x) < 0, \qquad x > x_0,$$

and since $Y(x_0) = Y'(x_0) = 0$, $h_1(x)$ is in $L(x_0, b)$; and since $I(1-\alpha; x_0, x_0^+|Y') = 0$,

$$Y'(x) = \int_{x_0}^{x} h_1(t)\, Y(x-t)\, dt < 0, \qquad x_0 < x < 2x_0,$$

which contradicts $Y(x) > 0$ for $x > x_0$ and $Y(x_0) = 0$.

These results may be summarized as follows:

THEOREM 3.1. *If* $0 < \alpha < 1$; $K_1 = 1$; $\lambda = 1$ *and* $a = 0$ *then the unique R-solution of* (*A*) *and* (*B*),

$$Y(x) = \sum_{q=1}^{\infty} \frac{(-1)^{q-1} x^{q\alpha-1}}{\Gamma(q\alpha)},$$

is positive for all $x > 0$.

COROLLARY 3.1.1. *If* $0 < \alpha < 1$ *then* $U_1(x-a; 1) = Y(x-a) > 0$ *for* $x > a$, *and any R-solution of* (*A*) *for* $h(x) = 0$ *has a zero on* $x > a$ *only if it is identically zero for* $x > a$.

We notice that if $\alpha = 1$, the corresponding solution is e^{-x} which is positive for $x > 0$. Also, $Y(x)$ satisfies the properties satisfied by $e^{-x}(\alpha = 1)$ given by the following theorem.

THEOREM 3.2. *Under the hypotheses of Theorem* 3.1, *we have*

(a) $\lim_{x\to\infty} Y(x) = 0$,

(b) $\int_{x_0}^{\infty} Y(t)\, dt = I(1-\alpha, 0, x_0|Y)$, *for* $x_0 > 0$, *and* $\int_0^{\infty} Y = 1$,

(c) if $0 < \beta < 1$, $\lim_{x\to\infty} I(\beta, 0, x|Y) = 0$.

Proof. Recall that $Y(x) + I(\alpha; 0, x|Y) = x^{\alpha-1}/\Gamma(\alpha)$ for $x > 0$. Since $0 < Y(x) < x^{\alpha-1}/\Gamma(\alpha)$, then part (a) follows and $I(\alpha; 0, x|Y) \to 0$ as $x \to \infty$. Now $D_x I(1-\alpha; 0, x|Y) = -Y(x) < 0$ and $I(1-\alpha; 0, x|Y) > 0$. Let

$$\lim_{x\to\infty} I(1-\alpha;\, 0, x|Y) = d \geqq 0.$$

Suppose that $d > 0$, then there exists a number $X > 0$ such that for $x > X$, $I(1-\alpha; 0, x|Y) > \frac{1}{2}d$ and also

$$I(1;\, 0, x|Y) > I(\alpha;\, X, x|I(1-\alpha;\, 0, t|Y)) > \frac{d(x-X)^{\alpha}}{2\Gamma(\alpha+1)} \to \infty \qquad \text{as } x \to \infty.$$

But $I(1-\alpha; 0, x|Y) + I(1; 0, x|Y) = 1$, so that we have a contradiction. Hence $d = 0$. Thus part (b) follows.

To prove part (c), let $0 < \beta < \alpha$. For $x > 1$, then

$$I(\beta;\, 0, x|Y) = \int_0^{x-1} Y(t)\, \frac{(x-t)^{\beta-1}}{\Gamma(\beta)}\, dt + \int_{x-1}^{x} Y(t)\, \frac{(x-t)^{\beta-1}}{\Gamma(\beta)}\, dt.$$

Since $x - t > 1$ and $(x-t)^{\beta-1} < (x-t)^{\alpha-1}$ for $0 < t < x-1$, then $I(\beta; 0, x|Y) < \Gamma(\alpha)/\Gamma(\beta)$ and

$$I(\alpha;\ 0, x|Y) + \frac{(x-1)^{\alpha-1}}{\Gamma(\alpha)\,\Gamma(\beta+1)} \to 0 \qquad \text{as } x \to \infty.$$

Similarly, it follows that if $0 < \beta < 1 - \alpha$, then $I(\beta; 0, x|Y) \to 0$ as $x \to \infty$. Furthermore

$$I(\alpha+\delta;\ 0, x|Y) + I(\delta;\ 0, x|Y) = \frac{x^{\alpha+\delta} - 1}{\Gamma(\alpha+\delta)}$$

from which we see that $I(\alpha + \delta; 0, x|Y) \to 0$ as $x \to \infty$ for $\alpha + \delta < 1$; and $\delta \leqslant \max(\alpha, 1 - \alpha)$. If

$$\beta > \max(\alpha, 1-\alpha) \geqslant \tfrac{1}{2} \text{ and } \delta = \beta - \alpha$$

then $0 < \delta < \frac{1}{2} \leqslant \max(\alpha, 1 - \alpha)$. Hence, the result is proved for all $0 < \beta < 1$.

To complete this discussion we call attention to the above properties for $\alpha = 1$ and the corresponding $Y(x) = e^{-x}$, together with a few properties of e^{zt}, where z is any number.

THEOREM 3.3. *If z is a complex number; $0 < \beta < 1$ and*

(a) *if* $Rz < 0$, *then* $\lim\limits_{x\to\infty} I(\beta;\ 0, x|e^{zt}) = 0$;

(b) *if* $Rz \geqslant 0$, *and* $z \neq 0$ *then* $\lim\limits_{x\to\infty} \left[I(\beta;\ 0, x|e^{zt}) - \frac{e^{zx}}{z^{\beta}} \right] = 0.$

Proof. We have

$$I(\beta;\ 0, x|e^{zt}) = \int_0^x e^{zt} \frac{(x-t)^{\beta-1}}{\Gamma(\beta)}\,dt = e^{zx} \int_0^x e^{-zu} \frac{u^{\beta-1}}{\Gamma(\beta)} du.$$

For $Rz \geqslant 0$, $z \neq 0$ we recognize the Laplace Transform

$$\int_0^\infty e^{-zu} u^{\beta-1} du = \Gamma(\beta) z^{-\beta}$$

from which part (b) follows immediately. If $Rz < 0$, then

$$||I(\beta;\ 0, x|e^{zt})|| \leqslant e^{Rzx} \int_0^x e^{-Rzu} \frac{u^{\beta-1}}{\Gamma(\beta)}\,du.$$

Let $\epsilon > 0$ and x_1 be a number such that

$$\frac{x_1^{\beta-1}}{\Gamma(\beta)} < -Rz \cdot \tfrac{1}{2}\epsilon, \quad \text{and} \quad e^{Rzx} \int_0^x e^{-Rzu} \frac{u^{\beta-1}}{\Gamma(\beta)}\,du < \tfrac{1}{2}\epsilon.$$

Then $\left\|I(\beta; 0, x|e^{zt})\right\| < \epsilon$, for $x > x_1$.

4. Behavior of the entire function $E_\alpha(z)$ on the real axis and its relation to the behavior of $Y(x)$. At the beginning of this century extensive studies were made of the entire function:

$$E_\alpha(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(p\alpha + 1)}; \qquad R\alpha > 0.$$

The following identities exist between the function $Y(x)$ of §3 and $E_\alpha(z)$:

THEOREM 4.1. *For the above mentioned* $E_\alpha(z)$ *and* $Y(x)$ *and* $\mathrm{R}\alpha > 0$ *and* $x > 0$:

(a) $I(1-\alpha;\, 0, x\,|\,Y) = E_\alpha(-x^\alpha)$,

(b) $\phi(x) = \int_0^x Y(t)\,dt = 1 - E_\alpha(-x^\alpha)$,

(c) $Y(x) = \alpha x^{\alpha-1}E'_\alpha(-x^\alpha)$.

Mittag-Leffler **(7)** proved that for $0 < \alpha < 2$; $|E_\alpha(z)| \to 0$ as $x \to \infty$ and $\frac{1}{2}\alpha\pi < \arg z < 2\pi - \frac{1}{2}\alpha\pi$; such a domain includes the negative real axis, thus

$$\lim_{x\to\infty} E_\alpha(-x) = 0, \qquad 0 < \alpha < 2.$$

By applying Theorem 3.2 (c) and Theorem 4.1 (a) we have another proof of this latter fact for $0 < \alpha < 1$. Furthermore, using Theorem 4.1 (c) we see that $E'_\alpha(x) > 0$ for $x < 0$. From the series form of $E_\alpha(x)$ we observe that $E'_\alpha(x) > 0$ and $E_\alpha(x) > 0$ for $x \geqslant 0$ and $0 < \alpha < 1$. Now, from Theorem 3.1, $Y(x) > 0$ and it follows immediately that $E_\alpha(-x^\alpha) > 0$ for $0 < \alpha < 1$ and $x > 0$. Also, from Theorem 4.1 (b) and the fact that $\phi(x) > 0$ we see that $E_\alpha(-x^\alpha) < 1$ for $x > 0$. These results are summarized in the following:

THEOREM 4.2. *For* $0 < \alpha < 1$, $E_\alpha(z)$ *has no zeros on the real axis;* $0 < E_\alpha(x) < 1$ *for* $x < 0$ *and* $E'_\alpha(x) > 0$ *for the whole real axis.*

Wiman **(12)** proved that for $0 < \alpha < 1$, the zeros of $E_\alpha(z)$ in the upper (or lower) half z-plane approach the line $\arg z = \frac{1}{2}\alpha\pi$ (or $-\frac{1}{2}\alpha\pi$) as the modulus of the zero increases without bound. However, his discussion will not supply the fact that there are no zeros on the negative real axis.

Now, consider $1 < \alpha < 2$ and reverse the roles of $Y(x)$ and $E_\alpha(x)$, that is, use $E_\alpha(x)$ to complete the picture of $Y(x)$. In addition to the previously mentioned result of Mittag-Leffler, we will make use of the fact due to Wiman **(12)** that for large x, $E_\alpha(-x) < 0$. Thus, using Theorem 4.1 (b) we see that for large x, $\phi(x) > 1$ and $\lim \phi(x) = 1$ as $x \to \infty$. Hence, for large x, $Y(x) < 0$ and $\lim Y(x) = 0$ as $x \to \infty$. Also, from Theorem 4.1 (c) and the fact that $Y(0) = 0$ it follows that $Y(x)$ has at least as many zeros on the non-negative x-axis as $E_\alpha(x)$ has on the negative x-axis. Summarizing these results together with one which is an immediate application of a result of Wiman we have:

THEOREM 4.3. *For* $1 < \alpha < 2$:

(a) $\lim Y(x) = 0$ *as* $x \to \infty$ *and* $Y(x) < 0$ *for* x *large*;

(b) $Y(x)$ *has a finite number of zeros for* $x \geqslant 0$ *and if for each* α, $N(\alpha)$ *is this number of zeros*

$$\lim_{\alpha\to 2} N(\alpha) = \infty.$$

By direct application of Theorem 4.1 (c) to another result of Wiman we have:

THEOREM 4.4. *For* $2 < \alpha$, $Y(x)$ *has infinitely many zeros, i.e.,* $Y(x)$ *is oscillatory on* $x \geqslant 0$.

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THE CAUCHY PROBLEM FOR A HYPERBOLIC SECOND ORDER EQUATION WITH DATA ON THE PARABOLIC LINE

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1. Introduction. In this paper we consider the Cauchy problem for the equation

$$(1)\quad h(x, y)\, K(y)\, v_{xx} - v_{yy} + a(x, y)\, v_x + b(x, y)\, v_y + c(x, y)\, v + f(x, y) = 0$$

with initial values prescribed on a segment of the x-axis. The coefficients in (1) are assumed to possess two continuous derivatives with respect to x and one continuous derivative with respect to y in the closure of the domain under consideration.[1] The function $K(y)$ is a monotone increasing function of y with $K(0) = 0$ and we suppose $h(x, y)$ is positive in the closure of the domain. Equation (1) is hyperbolic for positive values of y and parabolic on the line $y = 0$. The characteristics of (1) are given by the two families of curves

$$(2)\qquad \frac{dy}{dx} = \pm \frac{1}{\sqrt{Kh}}.$$

Frankl **(4)** solved the Cauchy problem for the equation

$$(3)\qquad y u_{xx} - u_{yy} + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u = 0$$

under the assumption that the coefficients $a(x, y)$, $b(x, y)$, and $c(x, y)$ are analytic. Berezin **(1)** treated the same problem for the equation

$$(4)\quad h(x, y)\, y^{\alpha}\, u_{xx} - u_{yy} + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u + f(x, y) = 0$$

with restrictions on the coefficients similar to those for (1), but with the condition $0 < \alpha < 2$. Starting from a different point of view Bers **(2)** solved the Cauchy problem for the equation

$$(5)\qquad K(y)\, u_{xx} - u_{yy} = 0$$

where $K(y)$ is a continuous monotone increasing function of y with $K(0) = 0$. A solution to the same problem has been obtained for equation (5) by Germain and Bader **(5)**. They make the additional assumption that $K(y) \sim cy$ as $y \to 0$ and thus make use of Riemann's method. The result of Bers shows that if the lower order terms are absent in an equation such as (4) there is no restriction on the rate of growth of the coefficient of u_{xx}. On the other hand

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[1]The smoothness conditions on the coefficients can be weakened slightly.

Berezin gives an example to show that for $\alpha > 2$ the Cauchy problem is not correctly set for equation (4). In solving the initial value problem for equation (1) we shall impose such conditions on the coefficients as to encompass (except for slight differences in smoothness requirements) the previous results on this problem.

Let D be the domain bounded by a segment $a_0 < x < a_1$ of the x-axis and the characteristics Γ_1 and Γ_2 of the families (2) emanating from $(a_0, 0)$ and $(a_1, 0)$ respectively, and which intersect. The initial values are given by two functions $\tau(x)$, $\nu(x)$, $a_0 < x < a_1$ which are assumed to have continuous fourth derivatives.[3] That is, we seek a solution of (1) in D satisfying the conditions

$$v(x, 0) = \tau(x), \quad v_y(x, 0) = \nu(x), \quad a_0 < x < a_1. \tag{6}$$

With the change of variable

$$w(x, y) = v(x, y) - y\nu(x) - \tau(x)$$

equation (1) takes the form

$$h(x, y)K(y)w_{xx} - w_{yy} + a(x, y)w_x + b(x, y)w_y + c(x, y)w + F(x, y) = 0 \tag{7}$$

where

$$F(x, y) = hK(y\nu'' + \tau'') + a(y\nu' + \tau') + b\nu + c(y\nu + \tau) + f.$$

The initial conditions (6) become

$$w(x, 0) = w_y(x, 0) = 0, \qquad a_0 < x < a_1. \tag{8}$$

We restrict our considerations to equation (7) and inquire under what circumstances the Cauchy problem is correctly set. We shall show that the Cauchy problem is indeed correctly set if the condition

$$\frac{y\, a(x, y)}{\sqrt{K(y)}} \to 0 \text{ as } y \to 0, \qquad a_0 < x < a_1, \tag{9}$$

is satisfied.

This condition is automatically fulfilled in the case of (5) while for equation (4) it makes no additional requirement on $a(x, y)$ if $0 < \alpha < 2$. On the other hand we find as a special case that for the equation

$$h(x, y)\, K(y)\, u_{xx} - u_{yy} + b(x, y)\, u_y + c(x, y)\, u + f(x, y) = 0$$

the Cauchy problem is correctly set for all monotone $K(y)$ as (9) is clearly satisfied in this case. This is an example of a result not obtainable from any of the previous works on the singular Cauchy problem.

THEOREM. *Assume that in the closure of D the coefficients of equation* (7) *are twice continuously differentiable with respect to x, once continuously differentiable*

[3] Assuming the third derivatives satisfy a Lipschitz condition would be sufficient.

with respect to y *and that* $h(x, y) > 0$. *Suppose* $K(y)$ *is a monotone increasing function of* y *with* $K(0) = 0$. *Then if condition* (9) *is satisfied the Cauchy problem for equation* (7) *is correctly set.*

In §2 equation (7) is transformed to a system of integral equations. The above theorem is proved in §3, and in §4 some remarks are made about more general equations.[3]

2. Reduction to a System of Integral Equations. We introduce the new unknown functions

$$u_1(x, y) = w(x, y),\ u_2(x, y) = \sqrt{Kh}\ w_x + w_y,\ u_3(x, y) = -\sqrt{Kh}\ w_x + w_y.$$

Then (7) may be written as the system

$$
\begin{aligned}
u_{1y} &= \tfrac{1}{2}(u_2 + u_3),\\
u_{2y} - \sqrt{Kh}\, u_{2x} &= cu_1 + \tfrac{1}{2}\left(\frac{a}{\sqrt{Kh}} + b + \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} - (\sqrt{Kh})_x\right)u_2\\
&\quad + \tfrac{1}{2}\left(-\frac{a}{\sqrt{Kh}} + b - \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} + (\sqrt{Kh})_x\right)u_3 + F(x, y),\\
u_{3y} + \sqrt{Kh}\, u_{3x} &= cu_1 + \tfrac{1}{2}\left(\frac{a}{\sqrt{Kh}} + b - \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} + (\sqrt{Kh})_x\right)u_2\\
&\quad + \tfrac{1}{2}\left(-\frac{a}{\sqrt{Kh}} + b + \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} + (\sqrt{Kh})_x\right)u_3 + F(x, y),
\end{aligned}
\tag{10}
$$

subject to the initial conditions

$$u_1(x, 0) = u_2(x, 0) = u_3(x, 0) = 0, \quad a_0 < x < a_1.$$

The characteristics of (10) are the lines $x =$ const. and the two families of curves given by (2). Let $P(x, y)$ be a point in D and construct the three characteristics of (10) passing through P. The left side of each of the equations in (10) represents a derivative in a characteristic direction. If we denote by s_2 the member of the family

$$\frac{dy}{dx} = \frac{-1}{\sqrt{Kh}}$$

passing through P and by s_3 the member of the family

$$\frac{dy}{dx} = \frac{1}{\sqrt{Kh}}$$

[3]The smoothness conditions on the coefficients can be weakened slightly.

passing through P we can write (10) in the form

$$\frac{du_1}{dy} = \tfrac{1}{2}(u_2 + u_3),$$

$$\frac{du_2}{ds_2} = \frac{c}{\sqrt{(1+Kh)}}u_1 + \frac{1}{2\sqrt{(1+Kh)}}\left(\frac{a}{\sqrt{Kh}} + b + \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} - (\sqrt{Kh})_x\right)u_2 + \frac{1}{2\sqrt{(1+Kh)}}\left(-\frac{a}{\sqrt{Kh}} + b - \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} + (\sqrt{Kh})_x\right)u_3 + \frac{F}{\sqrt{(1+Kh)}}$$

(11)

$$\frac{du_3}{ds_3} = \frac{c}{\sqrt{(1+Kh)}}u_1 + \frac{1}{2\sqrt{(1+Kh)}}\left(\frac{a}{\sqrt{Kh}} + b - \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} - (\sqrt{Kh})_x\right)u_2 + \frac{1}{2\sqrt{(1+Kh)}}\left(-\frac{a}{\sqrt{Kh}} + b + \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} + (\sqrt{Kh})_x\right)u_3 + \frac{F}{\sqrt{(1+Kh)}}.$$

To simplify the notation we define the quantities

$$A = \frac{c}{\sqrt{(1+Kh)}}, \quad B_2 = \frac{1}{2\sqrt{(1+Kh)}}\left(b - (\sqrt{Kh})_x + \frac{h_y}{2h}\right)$$

$$C_2 = \frac{1}{2\sqrt{(1+Kh)}}\left(b + (\sqrt{Kh})_x - \frac{h_y}{2h}\right),$$

$$D_2 = \frac{1}{2\sqrt{(1+Kh)}}\left(\frac{a}{\sqrt{Kh}} + \frac{K'}{2K}\right)$$

$$B_3 = \frac{1}{2\sqrt{(1+Kh)}}\left(b - (\sqrt{Kh})_x - \frac{h_y}{2h}\right),$$

$$C_3 = \frac{1}{2\sqrt{(1+Kh)}}\left(b + (\sqrt{Kh})_x + \frac{h_y}{2h}\right)$$

$$D_3 = \frac{1}{2\sqrt{(1+Kh)}}\left(\frac{a}{\sqrt{Kh}} - \frac{K'}{2K}\right), \quad E = \frac{F}{\sqrt{(1+Kh)}}.$$

The system (11) then becomes

$$(12)\qquad \begin{aligned} \frac{du_1}{dy} &= \tfrac{1}{2}(u_2 + u_3) \\ \frac{du_i}{ds_i} &= Au_1 + B_i u_2 + C_i u_3 + D_i(u_2 - u_3) + E \qquad (i = 2, 3). \end{aligned}$$

Integrating (12) along the characteristics we obtain the system of integral equations

$$(13)\qquad \begin{aligned} u_1(x, y) &= \tfrac{1}{2} \int_0^y [u_2(x, y) + u_3(x, y)]\, dy \\ u_i(x, y) &= \int_0^{s_i} [Au_1 + B_i u_2 + C_i u_3 + D_i(u_2 - u_3) + E]\, ds_i \quad (i = 2, 3). \end{aligned}$$

Any solution $u_1(x, y)$ of (13) with the proper differentiability properties will clearly satisfy (7).

3. Proof of Theorem. It suffices to prove the theorem for an arbitrarily small segment $0 \leq y \leq \eta$. For, once the solution is determined in such a domain the standard Cauchy problem may be solved on the line $y = \eta$ yielding the result in D. We select initially for $K(y)$ the function y^α, $\alpha > 0$, since the main argument of the proof is exhibited in this case. In the last paragraph of this section the case where $K(y)$ does not behave like y^α is discussed.

Let $P(\xi, \eta)$ be a point of D and suppose $x = x_2(y)$, $x = x_3(y)$ are the characteristics of (2) passing through P. We have the inequality for $0 \leq y \leq \eta$

$$(14)\qquad |x_2 - x_3| \leq 2 \int_0^y |\sqrt{Kh}|\, dy \leq My^{\frac{1}{2}\alpha+1}$$

where M is an upper bound in D for $4\sqrt{h}/(\alpha + 2)$. The quantity M will denote throughout a positive constant that dominates in absolute value the coefficients of (7) and their first derivatives with respect to x. That is, we require that M be so large that

$$(15)\qquad |A|, |A_x|, |B_i|, |B_{ix}|, |C_i|, |C_{ix}|, |h^{-1}|, |h_y h^{-1}|, |h_x h^{-1}|, |E|, |E_x| \leq M$$

for all x, y in D and $i = 2, 3$. From condition (9) we have

$$(16)\qquad a(x, y) \sim \delta(y)\, y^{\frac{1}{2}\alpha-1}$$

where $\delta(y) \to 0$ as $y \to 0$. We select $\gamma (0 < \gamma < 1)$ and $\eta (>0)$ so that

$$(17)\qquad \begin{aligned} &\frac{3}{2} M\eta + \left[\delta(\eta)\, M + \frac{\alpha}{2}\right] \frac{1}{\alpha + 2}\, \eta^{\frac{1}{2}\alpha} < \gamma, \\ &\frac{6\, M^2 \eta^2}{\alpha + 6} + \frac{8M\eta}{\alpha + 2} + \left[\delta(\eta) + \frac{\alpha}{2M}\right] \frac{2M}{\alpha + 2} < \gamma. \end{aligned}$$

It is easy to see that if γ is taken sufficiently close to 1 and η sufficiently small the inequalities (17) can always be satisfied.

To establish the existence of a solution of the system (13) we proceed by iterations. We define $u_i^{(0)}(x, y) \equiv 0$ $(i = 1, 2, 3)$, and the quantities $u_i^{(k)}(x, y)$ by the relations

$$u_1^{(k)} = \tfrac{1}{2}\int_0^y [u_2^{(k-1)} + u_3^{(k-1)}]\,dy$$

$$u_i^{(k)} = \int_0^{s_i} [Au_1^{(k-1)} + B_i u_2^{(k-1)} + C_i u_3^{(k-1)} + D_i(u_2^{(k-1)} - u_3^{(k-1)}) + E]\,ds_i \qquad (i = 2, 3).$$

We shall show that the sequences $\{u_i^{(k)}(x, y)\}$ $(i = 1, 2, 3)$ converge uniformly in that part of D contained in the strip $0 < y < \eta$. We first establish some inequalities for the $\{u_i^{(k)}(x, y)\}$. To do this it is necessary to examine the characteristics (2) and inequality (14). If $P(\xi_0, \eta_0)$ is a point of D with $\eta_0 < \eta$ then the characteristics through P are given by the solutions of (2) which we write in the form

$$x = x_2(y;\ \xi_0, \eta_0), \quad x = x_3(y;\ \xi_0, \eta_0).$$

Let D_1 be the domain bounded by these characteristics and the line $y = 0$. Then it is clear that an inequality such as (14) (with perhaps M somewhat larger) will hold for any two points $P_1(x_2, y)$ and $P_2(x_3, y)$ in D.

LEMMA 1. *For all k the inequalities*

$$(18)\qquad \begin{gathered} |u_i^{(k)}(x, y)| < M\sum_{j=0}^{k}\gamma^j y, \quad |u_i^{(k)}(x_2, y) - u_i^{(k)}(x_3, y)| < M\sum_{j=0}^{k}\gamma^j y^{\frac{1}{2}\alpha+1}, \\ |u_2^{(k)}(x, y) - u_3^{(k)}(x, y)| < M\sum_{j=0}^{k}\gamma^j y^{\frac{1}{2}\alpha+1} \qquad (i = 1, 2, 3). \end{gathered}$$

hold in D_1.

Proof. We proceed by induction, establishing all inequalities simultaneously. That is, we show that all inequalities (18) hold for $n = 1$, and then assuming they *all* hold for $n = k$ we establish each inequality for $n = k + 1$. Clearly $u_1^{(1)}(x, y)$ vanishes and

$$|u_i^{(1)}(x, y)| < \int_0^y |E|\,dy < My < M\sum_{j=0}^{1}\gamma^j y \qquad (i = 2, 3).$$

Further, for $i = 2, 3$

$$\begin{aligned} |u_i^{(1)}(x_2, y) - u_i^{(1)}(x_3, y)| &< \int_0^y |E(x_i(t; x_2, y), t) - E(x_i(t; x_3, y), t)|\,dt \\ &< \int_0^y |E_x||x_i(t; x_2, y) - x_i(t; x_3, y)|\,dt \\ &< M\int_0^y |x_2 - x_3|\,dy < M\int_0^y My^{\frac{1}{2}\alpha+1}\,dy \\ &< M\sum_{j=0}^{1}\gamma^j y^{\frac{1}{2}\alpha+1} \end{aligned}$$

and

$$|u_2^{(1)}(x, y) - u_3^{(1)}(x, y)| < \int_0^y |E(x_2, y) - E(x_3, y)|\, dy < M \sum_{j=0}^{1} \gamma^j y^{\frac{1}{2}\alpha+1}.$$

Assume the result holds for $n = k$. Then for $n = k + 1$ we find

$$|u_1^{(k+1)}(x, y)| < \tfrac{1}{2} \int_0^y [|u_2^{(k)}| + |u_3^{(k)}|]\, dy$$

$$< \tfrac{1}{2} \int_0^y 2M \sum_{j=0}^{k} \gamma^j y dy < M \sum_{j=0}^{k} \gamma^j \tfrac{1}{2} y^2 < M \sum_{j=0}^{k+1} \gamma^j y,$$

as we can add to (17) the condition that η be less than 2γ. For $i = 2, 3$ we obtain

$$|u_i^{(k+1)}(x, y)| < \int_0^y \Big\{ |A| \sum_{j=0}^{k} M\gamma^j y + |B_i| \sum_{j=0}^{k} M\gamma^j y + |C_i| \sum_{j=0}^{k} M\gamma^j y$$

$$+ |D_i| \sum_{j=0}^{k} M\gamma^j y^{\frac{1}{2}\alpha+1} + |E| \Big\}\, dy$$

$$< M \int_0^y \Big\{ 3M \sum_{j=0}^{k} \gamma^j y + \tfrac{1}{2}\Big(\frac{\delta(y)}{y|h|} + \frac{\alpha}{2y}\Big) y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j + 1 \Big\}\, dy$$

$$< My\Big[1 + \Big\{\frac{3}{2} My + \Big(\frac{\delta(y)}{|h|} + \frac{\alpha}{2}\Big) \frac{y^{\frac{1}{2}\alpha}}{\alpha + 2}\Big\} \sum_{j=0}^{k} \gamma^j\Big]$$

$$< M \sum_{j=0}^{k+1} \gamma^j y,$$

the last inequality being valid because of the first of the inequalities in (17). We also have

$$|u_2^{(k+1)}(x, y) - u_3^{(k+1)}(x, y)| < \int_0^y |A(x_2, y)\, u_1^{(k)}(x_2, y) - A(x_3, y)\, u_1^{(k)}(x_3, y)$$

$$+ B_2(x_2, y)\, u_2^{(k)}(x_2, y) - B_3(x_3, y)\, u_2^{(k)}(x_3, y) + C_2(x_2, y)\, u_3^{(k)}(x_2, y)$$

$$- C_3(x_3, y)\, u_3^{(k)}(x_3, y) + D_2(x_2, y)[u_2^{(k)}(x_2, y) - u_3^{(k)}(x_2, y)]$$

$$- D_3(x_3, y)[u_2^{(k)}(x_3, y) - u_3^{(k)}(x_3, y)] + E(x_2, y) - E(x_3, y)|\, dy.$$

To get a bound for the integral on the right side we have the following estimates

$$|A(x_2, y)\, u_1^{(k)}(x_2, y) - A(x_3, y)\, u_1^{(k)}(x_3, y)|$$

$$< |A(x_2, y)\, u_1^{(k)}(x_2, y) - A(x_3, y)\, u_1^{(k)}(x_2, y)|$$

$$+ |A(x_3, y)[u_1^{(k)}(x_2, y) - u_1^{(k)}(x_3, y)]|$$

$$< M \sum_{j=0}^{k} \gamma^j y |A(x_2, y) - A(x_3, y)| + M^2 \sum_{j=0}^{k} \gamma^j y^{\frac{1}{2}\alpha+1}.$$

Now applying the theorem of the mean to the first term on the right we find

$$
\begin{aligned}
&|A(x_2, y)\, u_1^{(k)}(x_2, y) - A(x_3, y)\, u_1^{(k)}(x_3, y)| \\
&\qquad\qquad < M^2 \sum_{j=0}^{k} \gamma^j y |x_2 - x_3| + M^2 \sum_{j=0}^{k} \gamma^j y^{\frac{1}{2}\alpha+1} \\
&\qquad\qquad < M^2 \sum_{j=0}^{k} \gamma^j y M y^{\frac{1}{2}\alpha+1} + M^2 \sum_{j=0}^{k} \gamma^j y^{\frac{1}{2}\alpha+1}.
\end{aligned}
$$

We also have the inequality

$$
\begin{aligned}
&|B_2(x_2, y)\, u_2^{(k)}(x_2, y) - B_3(x_3, y)\, u_2^{(k)}(x_3, y) \\
&\qquad\qquad + C_2(x_2, y)\, u_3^{(k)}(x_2, y) - C_3(x_3, y)\, u_3^{(k)}(x_3, y)| \\
&< |B_2(x_2, y) - B_2(x_3, y)||u_2^{(k)}(x_2, y)| \\
&+ |[B_2(x_3, y) - B_3(x_3, y)]\, u_2^{(k)}(x_2, y) \\
&\qquad\qquad + [C_2(x_3, y) - C_3(x_3, y)]\, u_3^{(k)}(x_2, y)| \\
&+ |B_3(x_3, y)||u_2^{(k)}(x_2, y) - u_2^{(k)}(x_3, y)| \\
&\qquad\qquad + |C_3(x_3, y)||u_3^{(k)}(x_2, y) - u_3^{(k)}(x_3, y)|.
\end{aligned}
\tag{19}
$$

Taking into account the definitions of B_2, B_3, C_2, C_3 in the second term on the right above, we obtain after an application of the theorem of the mean the following upper bound for the right side:

$$
\begin{aligned}
&M|x_2 - x_3| M \sum_{j=0}^{k} \gamma^j y + \left|\frac{h_y}{4h}\right| |u_2^{(k)}(x_2, y) - u_3^{(k)}(x_2, y)| \\
&\quad + M|u_2^{(k)}(x_2, y) - u_2^{(k)}(x_3, y)| + M|u_3^{(k)}(x_2, y) - u_3^{(k)}(x_3, y)| \\
&\quad < M^3 y^{\frac{1}{2}\alpha+2} \sum_{j=0}^{k} \gamma^j + 3M^2 y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&|u_2^{(k+1)}(x, y) - u_3^{(k+1)}(x, y)| < \int_0^y \Big\{ M^3 y^{\frac{1}{2}\alpha+2} \sum_{j=0}^{k} \gamma^j + M^2 y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j \\
&\quad + M^3 y^{\frac{1}{2}\alpha+2} \sum_{j=0}^{k} \gamma^j + 3M^2 y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j + |D_2| M y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j + |D_3| M y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + M^2 y^{\frac{1}{2}\alpha+1} \Big\}\, dy \\
&\quad < M y^{\frac{1}{2}\alpha+1} \left[\frac{2M}{\alpha+4} y + \left(\sum_{j=0}^{k} \gamma^j \right) \left\{ \frac{4M^2 y^2}{\alpha+6} + \frac{8My}{\alpha+2} + \left(\delta(y) M + \frac{\alpha}{2} \right) \frac{2}{\alpha+2} \right\} \right].
\end{aligned}
$$

And taking inequality (17) into account we finally find

$$
|u_2^{(k+1)}(x, y) - u_3^{(k+1)}(x, y)| < M y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k+1} \gamma^j.
$$

The proof for the cases

$$
|u_i^{(k+1)}(x_2, y) - u_i^{(k+1)}(x_3, y)| \qquad (i = 1, 2, 3)
$$

is completely analogous and may be omitted. The only change required is that for $i = 2, 3$ the inequality

$$\frac{6M^2y^2}{\alpha+6} + \frac{6My}{\alpha+2} + \left(M\delta(y) + \frac{\alpha}{2}\right)\frac{2}{\alpha+2} < \gamma$$

is employed. However this follows from (17) and the induction is complete.

LEMMA 2. *For all k the inequalities*

$$|u_i^{(k+1)}(x, y) - u_i^{(k)}(x, y)| < M\gamma^k y \qquad (i = 1, 2, 3)$$

$$(20) \quad |u_2^{(k+1)}(x, y) - u_3^{(k+1)}(x, y) - u_2^{(k)}(x, y) + u_3^{(k)}(x, y)| < M\gamma^k y^{\frac{1}{2}\alpha+1}$$

$$|u_i^{(k+1)}(x_2, y) - u_i^{(k+1)}(x_3, y) - u_i^{(k)}(x_2, y) + u_i^{(k)}(x_3, y)| < M\gamma^k y^{\frac{1}{2}\alpha+1}$$

$$(i = 1, 2, 3)$$

hold in D_1.

Proof. We proceed by induction. It is clear that each of the inequalities holds for $n = 1$. Assume they are valid for $n = k$. For $n = k + 1$ we have

$$\begin{aligned} |u_i^{(k+1)}(x, y) - u_i^{(k)}(x, y)| &< \int_0^y \Big\{ |A(x_2, y)||u_1^{(k)} - u_1^{(k-1)}| + |B_i||u_2^{(k)} - u_2^{(k-1)}| \\ &\qquad + |D_i||u_2^{(k)} - u_3^{(k)} - u_2^{(k-1)} + u_3^{(k-1)}| \Big\} dy \\ &< M\gamma^{k-1}\left[My^2 + \left(M\delta(y) + \frac{\alpha}{2}\right)\frac{1}{\alpha+2} y^{\frac{1}{2}\alpha+1} \right] < M\gamma^k y \end{aligned}$$

$$(i = 2, 3),$$

and similarly for $i = 1$. Also,

$$\begin{aligned} |u_2^{(k+1)}(x, y) &- u_3^{(k+1)}(x, y) - u_2^{(k)}(x, y) + u_3^{(k)}(x, y)| < \int_0^y |A(x_2, y)\, u_1^{(k)}(x, y) \\ &+ B_2(x_2, y) u_2^{(k)}(x_2, y) + C_2(x_2, y) u_3^{(k)}(x_2, y) + D_2(x_2, y)[u_2^{(k)}(x_2, y) - u_3^{(k)}(x_2, y)] \\ &- A(x_3, y)\, u_1^{(k)}(x_3, y) - B_3(x_3, y)\, u_2^{(k)}(x_3, y) \\ &- C_3(x_3, y)\, u_3^{(k)}(x_3, y) - D_3(x_3, y)[u_2^{(k)}(x_3, y) - u_3^{(k)}(x_3, y)] \\ &- A(x_2, y)\, u_1^{(k-1)}(x_2, y) - B_2(x_2, y)\, u_2^{(k-1)}(x_2, y) - C_2(x_2, y)\, u_3^{(k-1)}(x_2, y) \\ &- D_2(x_2, y)[u_2^{(k-1)}(x_2, y) - u_3^{(k-1)}(x_2, y)] + A(x_3, y)\, u_1^{(k-1)}(x_3, y) \\ &+ B_3(x_3, y)\, u_2^{(k-1)}(x_3, y) + C_3(x_3, y)\, u_3^{(k-1)}(x_3, y) \\ &+ D_3(x_3, y)[u_2^{(k-1)}(x_3, y) - u_3^{(k-1)}(x_3, y)| \, dy. \end{aligned}$$

We make the estimate

$$
\begin{aligned}
&|A(x_2, y)[u_1^{(k)}(x_2, y) - u_1^{(k-1)}(x_2, y)] - A(x_3, y)[u_1^{(k)}(x_3, y) - u_1^{(k-1)}(x_3, y)]\\
&\quad < |A(x_2, y) - A(x_3, y)|\,|u_1^{(k)}(x_2, y) - u_1^{(k-1)}(x_2, y)|\\
&\qquad + |A(x_3, y)|\,|u_1^{(k)}(x_2, y) - u_1^{(k-1)}(x_2, y) - u_1^{(k)}(x_3, y) + u_1^{(k-1)}(x_3, y)|\\
&\quad < M|x_2 - x_3|\, M\, \gamma^{k-1} y + M^2\, \gamma^{k-1} y^{\frac{1}{2}\alpha+1} < M\, \gamma^{k-1} y^{\frac{1}{2}\alpha+1}(My + M).
\end{aligned}
$$

Similar bounds are found for the remaining terms B_i, C_i, D_i. We have only to be sure to combine the terms involving B_2, B_3, C_2, C_3 as in the estimate (19). This yields the inequality

$$
\begin{aligned}
&|B_2(x_2, y)[u_2^{(k)}(x_2, y) - u_2^{(k-1)}(x_2, y)] - B_3(x_3, y)[u_2^{(k)}(x_3, y) - u_2^{(k-1)}(x_3, y)]\\
&\quad + C_2(x_2, y)[u_3^{(k)}(x_2, y) - u_3^{(k-1)}(x_2, y)] - C_3(x_3, y)[u_3^{(k)}(x_3, y) - u_3^{(k-1)}(x_3, y)|\\
&\quad < 2M^2\gamma^{k-1} y|x_2 - x_3| + \left|\frac{2h_y}{h}\right| M\, \gamma^{k-1} y^{\frac{1}{2}\alpha+1} + 2M^2\gamma^{k-1} y^{\frac{1}{2}\alpha+1}\\
&\quad < y^{\frac{1}{2}\alpha+1} M\, \gamma^{k-1}(2M^2 y + 3M).
\end{aligned}
$$

With the aid of these estimates inequalities (20) follow.

From Lemma 2 it is clear that the sequences $\{u_i^{(k)}(x, y)\}$ $(i = 1, 2, 3)$ converge uniformly. Since each $u_i^{(k)}(x, y)$ is continuous, so are the limits, which we denote by $u_i(x, y)$. Inequalities (18) yield

$$
\begin{aligned}
&|u_i(x, y)| < M_1 y, && (i = 1, 2, 3)\\
&|u_2(x, y) - u_3(x, y)| < M_1 y^{\frac{1}{2}\alpha+1} && (21)
\end{aligned}
$$

where

$$
M_1 = M \sum_{j=0}^{\infty} \gamma^j.
$$

The limit functions obtained with the aid of Lemma 2 are easily seen to satisfy the system of integral equations (13) and the initial conditions $u_1(x, 0) = u_2(x, 0) = u_3(x, 0) = 0$, $a_0 < x < a_1$.

The uniqueness of the solution follows from the fact that the difference of two solutions would have to satisfy the homogeneous system

$$
\begin{aligned}
v_1 &= \tfrac{1}{2}\int_0^y (v_2 + v_3)\, dy,\\
v_i &= \int_0^{s_i} \{Av_1 + B_i v_2 + C_i v_3 + D_i(v_2 - v_3)\}\, ds_i \qquad (i = 2, 3).
\end{aligned}
$$

The functions v_i satisfy the inequalities (20) and repeated insertion of these in the right side above shows that each v_i must satisfy an inequality of the form $|v_i| < M_2\gamma^k$ for arbitrary k. Hence $v_i \equiv 0 (i = 1, 2, 3)$.

It remains to be shown that $w(x, y) = u_1(x, y)$ satisfies equation (7) and depends continuously on the given data. From the relation $u_{1y}(x, y) = \frac{1}{2}(u_2 + u_3)$ we see that w possesses a derivative with respect to y. Also,

$$w_x = \frac{u_2 - u_3}{2\sqrt{Kh}},$$

and from the basic inequality for $u_2 - u_3$ it is clear that w_x exists for $y > 0$. To obtain the existence of the second derivatives of w we consider the system of integral equations

$$u_{1x} = \tfrac{1}{2}\int_0^{y_0} (u_{2x} + u_{3x})\,dy$$

(22)
$$u_{ix}(x_0, y_0) = \int_0^{y_0} \{A u_{1x} + B_i u_{2x} + C_i u_{3x} + D_i(u_{2x} - u_{3x}) + E_x + A_x u_1 + B_{ix} u_2 + C_{ix} u_3 + D_{ix}(u_2 - u_3)\} \frac{d\gamma_i}{dx_0}\,dy, \qquad (i = 2, 3)$$

where $y = \gamma_i(x; x_0, y_0)$ $(i = 2, 3)$ are the equations of the characteristics through $P(x_0, y_0)$. The above system is obtained from (13) by differentiation with respect to x. An iteration process can be set up and a solution found by the same method employed in solving (13). It is in the solution of this system that the bounds for the second derivatives of the coefficients are employed. The solution of (22) yields the existence of the second derivatives of w. Since w satisfies (13) and has the required differentiability properties, it is the solution of (7) satisfying initial conditions (8). The continuous dependence on the given data follows at once from inequalities (21).

If $K(y)$ tends to zero more rapidly than any power of y we have the inequality

$$|x_2 - x_3| < 2\int_0^y \sqrt{Kh}\,dy < \theta(y)\sqrt{K}\,y,$$

where $\theta(y) \to 0$ as $y \to 0$. This is easily seen by considering the ratio

$$\int_0^y \sqrt{Kh}\,dy/\sqrt{K}\,y,$$

and noting that this approaches zero as $y \to 0$. Hence the estimate for $|x_2 - x_3|$, which is basic, is better in this case than the case $K(y) \sim y^\alpha$. Should $K(y) \to 0$ slower than any power of y, the argument used for the case $0 < \alpha < 2$ applies and condition (9) is unnecessary.

4. Other Equations. Conti (3) has shown that the Cauchy problem for the equation

(23)
$$h(x, y)y^\alpha u_{xx} - u_{yy} = f(x, y, u, u_x, u_y)$$

is correctly set for the range $0 < \alpha < 2$. The discussion of equation (7) can be modified to include equation (23). In this case condition (9) is replaced by the condition

$$\frac{y f_{u_y}(x, y, u, u_x, u_y)}{\sqrt{K}} \to 0, \quad \text{as } y \to 0, \quad a_0 < x < a_1,$$

and otherwise the arguments are analogous.

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AN EXPANSION THEOREM FOR A PAIR OF SINGULAR FIRST ORDER EQUATIONS

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1. Introduction. Titchmarsh **(4)** has shown how the classical method of complex variables can be used to obtain expansion theorems for the singular cases of the second order equation

$$y''(x) + [\lambda - q(x)]y(x) = 0. \tag{1}$$

The purpose of this paper is to indicate how these results can be generalized to the singular cases of the pair of first order equations

$$\begin{aligned} u'(x) - [\lambda + q_1(x)]v &= 0, \\ v'(x) + [\lambda + q_2(x)]u &= 0. \end{aligned} \tag{2}$$

The system (2) is a special case of the Dirac wave equations for a particle in a central field in the relativistic case, a system which has recently been investigated at the Oak Ridge National Laboratories. The presentation is largely formal to avoid excessive detail (see especially §4), but all omitted proofs are included in a report **(1)** and in any case are direct generalizations of the corresponding proofs given by Titchmarsh for the second order equation (1). The principal result may be summarized in the following

THEOREM I. *Consider the system* (2) *over the semi-infinite interval* $[0 \leqslant x < \infty]$ *and under the boundary condition*

$$u(0)\cos\alpha + v(0)\sin\alpha = 0, \tag{3}$$

where α *is a real constant. Let* $q_1(x)$, $q_2(x)$ *be real-valued continuous functions of* x *which belong to* $L(0, \infty)$. *We define a solution of* (2), (3) *as a pair of functions* $[u(x, \lambda), v(x, \lambda)]$, *with continuous first derivatives, satisfying this system. Then the values of* λ *for which such solutions exist form a continuous spectrum over the real* λ*-axis* $[-\infty < \lambda < \infty]$. *An arbitrary function pair* $f(x) = [f_1(x), f_2(x)]$ *which are continuous, of bounded variation and* $L^2(0, \infty)$, *and which satisfy the condition* (3) *at* $x = 0$ *may be represented by the generalized Fourier integrals*

$$f_1(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(\lambda)\, u(x, \lambda)\, d\lambda,$$

$$f_2(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(\lambda)\, v(x, \lambda)\, d\lambda,$$

where

$$g(\lambda) = [\mu^2(\lambda) + \nu^2(\lambda)]^{-1}\int_0^{\infty}[u(y, \lambda) f_1(y) + v(y, \lambda) f_2(y)]\, dy,$$

and $\mu(\lambda)$, $\nu(\lambda)$ *are functions of* λ *which do not vanish simultaneously.*

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2. Preliminaries. Consider the system (2) over the finite interval $[0 < x < b)$. Let

$$\phi(x, \lambda) = [\phi_1(x, \lambda), \phi_2(x, \lambda)], \qquad \theta(x, \lambda) = [\theta_1(x, \lambda), \theta_2(x, \lambda)]$$

be two solutions of (2) such that

$$\phi_1(0) = -\sin\alpha, \qquad \phi_2(0) = \cos\alpha,$$
$$\theta_1(0) = -\cos\alpha, \qquad \theta_2(0) = -\sin\alpha$$

Let the Wronskian of ϕ, θ be defined as $W_x[\phi, \theta] = \phi_1\theta_2 - \phi_2\theta_1$. Then it is easily shown that $W_x[\phi, \theta]$ is independent of x. Now $W_0[\phi, \theta] = 1$, so $\phi(x, \lambda)$ and $\theta(x, \lambda)$ are linearly independent solutions. A general solution of (2) may be written $\theta(x, \lambda) + l(\lambda)\, \phi(x, \lambda)$. If this general solution is required to satisfy a real boundary condition of Sturmian type at $x = b$, it is known **(3)** that the eigenvalues are real, simple, discrete, and extend from $\lambda = -\infty$ to $\lambda = +\infty$. Moreover, the corresponding eigenfunctions are real. To obtain the spectrum in the singular case we take the limit of the general solution as $b \to \infty$. Then, following Titchmarsh, it is easily shown that, for values of λ other than real values, (2) has a solution $\psi(x, \lambda) = [\psi_1, \psi_2]$, say

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\, \phi(x, \lambda), \tag{4}$$

which belongs to $L^2(0, \infty)$. The definition of the function $m(\lambda)$ depends upon a limit of circles in the complex λ-plane which may be either a limit point or a limit circle. In the limit circle case all solutions are $L^2(0, \lambda)$. In addition $m(\lambda)$ is analytic in either the upper or lower half plane, it has the property $m(\bar{\lambda}) = \overline{m(\lambda)}$, and its imaginary part determines the spectrum. We proceed to determine $m(\lambda)$ for our system.

3. Nature of the spectrum. In this section we investigate order properties of the solution of (2) for large values of x, and apply these properties to the determination of the spectrum. It can be verified directly that a solution of (2) where $\phi_1(0) = -\sin\alpha$, $\phi_2(0) = \cos\alpha$ satisfies

$$\phi_1(x, \lambda) = \sin(\lambda x - \alpha) + \int_0^x \phi_2 q_1 \cos\lambda(x-s)\, ds - \int_0^x \phi_1 q_2 \sin\lambda(x-s)\, ds, \tag{5}$$

$$\phi_2(x, \lambda) = \cos(\lambda x - \alpha) - \int_0^x \phi_2 q_1 \sin\lambda(x-s)\, ds - \int_0^x \phi_1 q_2 \cos\lambda(x-s)\, ds. \tag{6}$$

Let $\lambda = \sigma + it$, $t > 0$, and let $\phi_1(x, \lambda) = e^{tx} h_1(x)$, $\phi_2(x, \lambda) = e^{tx} h_2(x)$, substitute in (5), (6) and take absolute values. We obtain

$$\begin{aligned} |h_1(x)| &< M + \int_0^x \{|h_2|\cdot|q_1| + |h_1|\cdot|q_2|\}\, ds, \\ |h_2(x)| &< M + \int_0^x \{|h_2|\cdot|q_1| + |h_1|\cdot|q_2|\}\, ds, \end{aligned} \tag{7}$$

where $M = O(1)$ for large x. At this point we need the following lemma which is proved in another paper by the authors **(2)**:

LEMMA. *Let h_1, h_2, g_1, g_2 be non-negative functions of x over the interval* $[0 < x < x_1]$; *let h_1, h_2 be continuous and g_1, g_2 integrable over this interval. If h_1, h_2 satisfy the inequalities*

$$h_1(x), h_2(x) < M + \int_0^x \{h_1(s)\, g_1(s) + h_2(s)\, g_2(s)\}\, ds,$$

then

$$h_1(x), h_2(x) < C_1 \exp\left\{ \int_0^x (g_1 + g_2)\, ds \right\}. \qquad [0 < x < x_1].$$

This lemma may be applied to the inequalities (7) to yield the result

$$|h_1|, |h_2| < M \exp\left\{ \int_0^x [|q_1| + |q_2|]\, ds \right\},$$

and since $q_1(x)$, $q_2(x)$ are $L(0, \infty)$ it follows that $h_1(x)$, $h_2(x)$ are bounded for all x. Hence for large x, $\phi_1(x, \lambda) = O(e^{tx})$, $\phi_2(x, \lambda) = O(e^{tx})$.

Now for real λ as $x \to \infty$ (5) and (6) may be written

$$\begin{aligned} \phi_1(x, \lambda) &= \mu(\lambda) \cos \lambda x + \nu(\lambda) \sin \lambda x + o(1), \\ \phi_2(x, \lambda) &= \mu(\lambda) \cos \lambda x - \nu(\lambda) \sin \lambda x + o(1), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mu(\lambda) &= -\sin \alpha + \int_0^\infty [q_1\phi_2 \cos \lambda s + q_2\phi_1 \sin \lambda s]\, ds, \\ \nu(\lambda) &= \cos \alpha + \int_0^\infty [q_1\phi_2 \sin \lambda s - q_2\phi_1 \cos \lambda s]\, ds, \end{aligned} \tag{9}$$

and $o(1)$ indicates terms which approach zero as $x \to \infty$. The integrals in (9) converge uniformly in λ and hence μ, ν are continuous and bounded functions of λ. If $\theta(x, \lambda)$ is that solution of (2) satisfying $\theta_1(0) = -\cos \alpha$, $\theta_2(0) = -\sin \alpha$, we may similarly write

$$\begin{aligned} \theta_1(x, \lambda) &= \xi(\lambda) \cos \lambda x + \eta(\lambda) \sin \lambda x + o(1), \\ \theta_2(x, \lambda) &= \eta(\lambda) \cos \lambda x - \xi(\lambda) \sin \lambda x + o(1), \end{aligned} \tag{10}$$

where

$$\begin{aligned} \xi(\lambda) &= -\cos \alpha + \int_0^\infty [q_1\theta_2 \cos \lambda\, s + q_2\theta_1 \sin \lambda\, s]\, ds, \\ \eta(\lambda) &= -\sin \alpha + \int_0^\infty [q_1\theta_2 \sin \lambda\, s - q_2\theta_1 \cos \lambda\, s]\, ds. \end{aligned} \tag{11}$$

Hence we have

$$W[\phi, \theta] = \phi_1\theta_2 - \phi_2\theta_1 = \mu\eta - \nu\xi + o(1).$$

But from the boundary conditions at $x = 0$ we know that $W[\phi, \theta] = 1$, so that for real λ as $x \to \infty$

$$\mu\eta - \nu\xi = 1. \tag{12}$$

We deduce from (12) that μ, ν cannot both vanish for the same λ.

Now for complex λ we can show by making use of the order properties on the solutions ϕ, θ and by the fact that $q_1(x)$, $q_2(x)$ are $L(0, \infty)$ that

$$\phi_1(x, \lambda) = e^{-\mathfrak{I}\lambda x}\,[M_1(\lambda) + o(1)], \tag{13}$$
$$\phi_2(x, \lambda) = e^{-\mathfrak{I}\lambda x}\,[M_2(\lambda) + o(1)], \tag{14}$$
$$\theta_1(x, \lambda) = e^{-\mathfrak{I}\lambda x}\,[N_1(\lambda) + o(1)], \tag{15}$$
$$\theta_2(x, \lambda) = e^{-\mathfrak{I}\lambda x}\,[N_2(\lambda) + o(1)], \tag{16}$$

where

$$\begin{aligned}
M_2(\lambda) &= -\frac{\sin\alpha}{2i} + \frac{\cos\alpha}{2} - \tfrac{1}{2}\int_0^\infty e^{\mathfrak{I}\lambda s}[iq_1\phi_2 + q_2\phi_1]\,ds,\\
M_1(\lambda) &= -\frac{\sin\alpha}{2} - \frac{\cos\alpha}{2i} + \tfrac{1}{2}\int_0^\infty e^{\mathfrak{I}\lambda s}[q_1\phi_2 - iq_2\phi_1]\,ds,\\
N_1(\lambda) &= -\frac{\cos\alpha}{2} + \frac{\sin\alpha}{2i} + \tfrac{1}{2}\int_0^\infty e^{\mathfrak{I}\lambda s}[q_1\theta_2 - iq_2\theta_1]\,ds,\\
N_2(\lambda) &= -\frac{\sin\alpha}{2} - \frac{\cos\alpha}{2i} - \tfrac{1}{2}\int_0^\infty e^{\mathfrak{I}\lambda s}[iq_1\theta_2 + q_2\theta_1]\,ds.
\end{aligned} \tag{17}$$

Now let $\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\,\phi(x, \lambda)$ be that solution of (2) which for complex λ is $L^2(0, \infty)$. Using (13)-(16) we have

$$\psi_1 = \theta_1 + m\phi_1 = e^{-\mathfrak{I}\lambda x}\,[N_1 + mM_1 + o(1)],$$
$$\psi_2 = \theta_2 + m\phi_2 = e^{-\mathfrak{I}\lambda x}\,[N_2 + mM_2 + o(1)].$$

Now ϕ, θ certainly do not belong to $L^2(0, \infty)$, and if ψ is to be $L^2(0, \infty)$ we must have

$$m(\lambda) = -\frac{N_1}{M_1} = -\frac{N_2}{M_2}.$$

In (17) let λ tend to a real limit formally, i.e., let $t \to 0$. We obtain

$$N_1 \to \tfrac{1}{2}(\xi + i\eta), \qquad M_1 \to \tfrac{1}{2}(\mu + i\nu),$$
$$N_2 \to \tfrac{1}{2}(\eta - i\xi), \qquad M_2 \to \tfrac{1}{2}(\nu - i\mu),$$

where ξ, η, μ, ν are defined by (9), (11). Hence

$$\lim_{t\to 0} m(\lambda) = -\frac{\xi(\lambda) + i\eta(\lambda)}{\mu(\lambda) + i\nu(\lambda)}.$$

The imaginary part of $m(\lambda)$ for real λ is therefore

$$\mathfrak{I}\{m(\lambda)\} = -(\mu^2 + \nu^2)^{-1}, \tag{18}$$

and from (12), (18) it is apparent that $\mathfrak{I}\{m(\lambda)\}$ is a non-positive, non-vanishing continuous function bounded for all λ over the range $[-\infty < \lambda < \infty]$.

4. The expansion theorem. We define a function pair $\Phi(x, \lambda) = [\Phi_1, \Phi_2]$ by the equations

$$\Phi_1 = \psi_1(x, \lambda)\int_0^x \phi(y, \lambda)\cdot f(y)\,dy + \phi_1(x, \lambda)\int_x^\infty \psi(y, \lambda)\cdot f(y)\,dy,$$

$$\Phi_2 = \psi_2(x, \lambda)\int_0^x \phi(y, \lambda)\cdot f(y)\,dy + \phi_2(x, \lambda)\int_x^\infty \psi(y, \lambda)\cdot f(y)\,dy,$$

where λ is a complex parameter, $\phi(x, \lambda)$, $\theta(x, \lambda)$ are those solutions of (2) discussed in §3, $f(x) = [f_1(x), f_2(x)]$ is a pair of functions which are continuous and of bounded variation and which belong to $L^2(0, \infty)$, and, for example, $\phi \cdot f \equiv \phi_1 f_1 + \phi_2 f_2$. It may be verified directly that Φ_1, Φ_2 satisfy the inhomogeneous equations

$$\Phi_1' - [\lambda + q_1(x)]\Phi_2 = f_2(x),$$
$$\Phi_2' + [\lambda + q_2(x)]\Phi_1 = f_1(x).$$

It can be shown by deforming the straight line joining $-R + i\delta$ to $R + i\delta$ into the semicircle on that base and lying in the upper half plane that

$$f(x) = \lim_{R\to\infty} \left\{ -\frac{1}{i\pi} \int_{-R+i\delta}^{R+i\delta} \Phi(x, \lambda)\, d\lambda \right\}, \tag{19}$$

uniformly in $\delta > 0$. The proof of this is straightforward provided that the following theorem on the asymptotic behavior of the solutions of (2) for large λ is available.

THEOREM II. *Under the initial condition* $u(0) = -\sin\alpha$, $v(0) = \cos\alpha$ *the system* (2) *has the following asymptotic solution for large* $\lambda = \sigma + it$, $t > 0$,

$$u(x) = \sin(\xi - \alpha) + O(e^{tx}/|\lambda|),$$
$$v(x) = \cos(\xi - \alpha) + O(e^{tx}/|\lambda|),$$

where

$$\xi(x) = \lambda x + \tfrac{1}{2} \int_0^x [q_1(s) + q_2(s)]\, ds.$$

The proof of Theorem II is given in §5. Let us proceed to the limit formally in (19), recalling that $\psi = \theta + m\phi$, that

$$\Im\{m(\lambda)\} = -(\mu^2 + \nu^2)^{-1},$$

and that ϕ, θ are real for real λ. Letting $R \to \infty$, $\delta \to 0$, we obtain

$$\begin{aligned}
&\Im\left\{-\frac{1}{\pi}\int_{-R+i\delta}^{R+i\delta} \Phi_1(x, \lambda)\, d\lambda\right\} \\
&\quad = \Im\left\{-\frac{1}{\pi}\int_{-R+i\delta}^{R+i\delta} \psi_1(x, \lambda)\, d\lambda \int_0^x \phi\cdot f\, dy\right\} \\
&\qquad\qquad + \Im\left\{-\frac{1}{\pi}\int_{-R+i\delta}^{R+i\delta} \phi_1(x, \lambda)\, d\lambda \int_x^\infty \psi\cdot f\, dy\right\} \\
&\quad \to \frac{1}{\pi}\int_{-\infty}^{\infty} \phi_1(x, \lambda)(\mu^2 + \nu^2)^{-1} d\lambda \int_0^x \phi\cdot f\, dy \\
&\qquad\qquad + \frac{1}{\pi}\int_{-\infty}^{\infty} \phi_1(x, \lambda)\, d\lambda \int_x^\infty (\mu^2 + \nu^2)^{-1} \phi\cdot f\, dy.
\end{aligned}$$

The real part is non-contributing and hence upon combining the last two terms above we have the expansion for $f_1(x)$:

$$f_1(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi_1(x, \lambda)}{\mu^2 + \nu^2} d\lambda \int_0^\infty \{\phi_1(y, \lambda) f_1(y) + \phi_2(y, \lambda) f_2(y)\}\, dy.$$

The expansion for $f_2(x)$ is found similarly. The functions $f_1(x), f_2(x)$ are not entirely independent but must satisfy at $x = 0$ the condition

$$\cos\alpha\, f_1(0) + \sin\alpha\, f_2(0) = 0.$$

As a simple example of the expansion theorem consider the system (2) with $q_1(x) \equiv q_2(x) \equiv 0$. The solution of (2) for $u(0) = -\sin\alpha$, $v(0) = \cos\alpha$ is

$$\phi_1(x, \lambda) = \sin(\lambda x - \alpha), \qquad \phi_2(x, \lambda) = \cos(\lambda x - \alpha).$$

From §3 we have $\mu = -\sin\alpha$, $\nu = \cos\alpha$, $\mathfrak{I}\{m(\lambda)\} = -1$. The expansions for $f = [f_1(x), f_2(x)]$ are

$$f_1(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sin(\lambda x-\alpha)\, d\lambda \int_0^{\infty} \{\sin(\lambda y-\alpha) f_1(y) + \cos(\lambda y-\alpha) f_2(y)\} dy, \tag{20}$$

$$f_2(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \cos(\lambda x-\alpha)\, d\lambda \int_0^{\infty} \{\sin(\lambda y-\alpha) f_1(y) + \cos(\lambda y-\alpha) f_2(y)\} dy \tag{21}$$

To obtain the ordinary Fourier sine integral from these, set $\alpha = 0$, $f_2(x) \equiv 0$. Then using the fact that the integrand in (20) is an even function of λ we obtain

$$f_1(x) = \frac{2}{\pi}\int_0^{\infty} \sin\lambda x\, d\lambda \int_0^{\infty} \sin\lambda y\, f_1(y)\, dy.$$

5. An asymptotic solution for large λ. The proof of Theorem II may be outlined as follows. W. Hurwitz **(3)** has obtained a similar asymptotic solution to the system (2) but for real λ and in the regular case. We introduce functions $U(x, \lambda)$, $V(x, \lambda)$ by the relations

$$\begin{aligned} u(x) &= U + (1 + q_1/2\lambda)\sin(\xi - \alpha), \\ v(x) &= V + (1 + q_2/2\lambda)\cos(\xi - \alpha), \end{aligned} \tag{22}$$

where $\xi(x)$ is defined as in Theorem II. We wish to show that U, V are $O(e^{tx}/|\lambda|)$. Substitute into equation (2) and rearrange it to obtain

$$\begin{aligned} U'(x) - [\lambda + q_1]V &= P(x, \lambda)/\lambda, \\ V'(x) + [\lambda + q_2]U &= Q(x, \lambda)/\lambda, \end{aligned} \tag{23}$$

where

$$\begin{aligned} P(x, \lambda) &= C_1(x)\sin\xi + C_2(x)\cos\xi, \\ Q(x, \lambda) &= C_3(x)\cos\xi + C_4(x)\sin\xi, \end{aligned}$$

and the coefficients C_1, C_2, C_3, C_4 are continuous functions of x, independent of λ. Equations (23) may be written as integral equations

$$\begin{aligned} U(x, \lambda) &= F(x, \lambda)/\lambda + \int_0^x [-q_2 U \sin\lambda(x - s) + q_1 V \cos\lambda(x - s)]\, ds, \\ V(x, \lambda) &= G(x, \lambda)/\lambda + \int_0^x [-q_2 U \cos\lambda(x - s) + q_1 V \sin\lambda(x - s)]\, ds, \end{aligned} \tag{24}$$

where F and G are functions which for large λ can be shown to be $O(e^{tx})$. Now

set $U = e^{tx} U_1(x, \lambda)$, $V = e^{tx} V_1(x, \lambda)$, substitute into (24) and take absolute values. We obtain the inequalities

$$|U_1|,\ |V_1| < O(1/|\lambda|) + \int_0^x [|q_2|\cdot|U_1| + |q_1|\cdot|V_1|]\,ds.$$

Now since U_1, V_1 are continuous functions of x for all x and since q_1, q_2 are $L(0, \infty)$, the lemma in §3 applies and hence

$$|U_1|,\ |V_1| < O(1/|\lambda|) \exp \int_0^x (|q_1| + |q_2|)\,ds.$$

Thus U_1, V_1 are $O(1/|\lambda|)$ and U, V are $O(e^{tx}/|\lambda|)$ for each x over the interval $[0 < x < \infty]$. Theorem II follows immediately upon substituting these asymptotic expressions for U, V into relations (22).

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ON LINEAR PERTURBATION OF NON-LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. In the theory of the asymptotic solution or stability of ordinary differential equations most attention has been given to linear or nearly-linear cases. Investigations in this field, starting primarily with those of Kneser **(7)** on the equation $y'' + f(x)y = 0$, have by now mostly been summed up in results on the vector-matrix system $d\mathbf{y}/dx = \mathbf{A}\mathbf{y} + \mathbf{f}(\mathbf{y}, x)$, where $\mathbf{y}$ and $\mathbf{f}$ denote n-vectors of functions, and $\mathbf{A}$ an n-by-n matrix, frequently assumed constant. In the strictly linear case **(4; 8; 9)**, where $\mathbf{f}(\mathbf{y}, x) = \mathbf{B}(x)\mathbf{y}$, it is shown that with restrictions on $\mathbf{B}(x)$ as $x \to \infty$ all the solutions behave for large x as solutions of $d\mathbf{y}/dx = \mathbf{A}\mathbf{y}$. In the nearly-linear case however **(6; 10; 12)**, where we have restrictions on the magnitude of $||\mathbf{f}(\mathbf{y}, x)||/||\mathbf{y}||$ in some bounded $\mathbf{y}$-region, we may expect more than one type of solution; those for which $||\mathbf{y}(0)||$ is sufficiently small may be expected to behave asymptotically as solutions of $d\mathbf{y}/dx = \mathbf{A}\mathbf{y}$, while "larger" solutions may perhaps exhibit an entirely different behaviour.

In this paper I compare, in a special case, the single differential equations

$$y'' + y^{2n-1} = 0, \tag{1.1}$$

$$y'' + y^{2n-1} + h(x, y) = 0, \tag{1.2}$$

where the perturbing function $h(x, y)$ is in some sense small, and n is a positive integer. The cases in which $h(x, y)$ is, as a function of y, of the same or higher degree than y^{2n-1} are analogous to the above-mentioned linear and nearly-linear cases, respectively, and we may expect that all, or possibly only the "smaller," solutions of 1.2 will behave asymptotically as solutions of 1.1. In the non-linear case, $n > 1$, the possibility naturally presents itself that $h(x, y)$ might be of lower degree than y^{2n-1}, and here the situation may be expected to be just the opposite of that in the nearly-linear case, in that the "larger" solutions of 1.2 should behave as solutions of the unperturbed equation, while there may be (though there need not be) "smaller" solutions behaving differently.

My aim here is to make the latter considerations rigorous for the equation

$$y'' + y^{2n-1} + g(x)y = 0 \tag{1.3}$$

where $n > 2$ and $g(x)$ is suitably smooth and is small for large x. Roughly speaking, the situation may be summarised by saying that 1.3 has under fairly general conditions solutions behaving as solutions of 1.1 in respect of magni-

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tude and oscillatory behaviour. It may in addition have non-oscillatory solutions which are $o(1)$ for large x, particularly it seems if $g(x)$ is negative and small, but not too small, for large x. I conclude by establishing some conditions under which solutions of 1.3 can be actually approximated to in terms of solutions of 1.1; this presents slightly greater difficulties than in the linear case, since in the non-linear case the amplitude of the oscillations affects their frequency.

The equation 1.3 has an additional interest in that it may be regarded as a canonical form by transformation of $y'' + f(x)y^{2n-1} = 0$. Equations of both forms have interest in astrophysics, and special cases have been studied by Fowler **(5)**; oscillation criteria for the latter form have recently been given by me **(3)**.

2. Classification of solutions. This, like most of the subsequent working, is based on an adaptation of the polar-coordinate method. I define the amplitude-variable r of a solution y of 1.3, not identically zero, by

2.1 $$r^{2n} = y^{2n} + n(y')^2, \qquad r > 0,$$

so that for 1.1 r would be constant. The phase-variable θ will be defined later.

I show, in §3, that $r(x)$ tends under fairly wide conditions to a constant value as $x \to \infty$. Solutions for which $r(\infty)$ exists and is positive I term of type I, those for which $r(x) \to 0$ of type II; these are respectively the oscillatory and the $o(1)$ solutions referred to in §1.

While the type I solutions are all of the same character, the type II solutions, if they exist at all, need not form a homogeneous set. A possible subdivision would be according to the relative magnitude of the three terms on the left of 1.3, into type II(a) for which the first and third terms predominate, type II(b) for which the second and third terms predominate, and type II(c) for which all three terms are of the same order of magnitude. In this paper however I consider only type II solutions as a whole, irrespective of any subclassification.

These classifications may be illustrated in the case of

2.2 $$y'' + y^3 - ay/x^2 = 0,$$

where a is a real constant. For any a there is a two-parameter family of type I solutions, while type II solutions exist only for $a > 2$. If $a > 2$ we have precisely two solutions of type II(c), given by

$$y = \pm \sqrt{a-2}\, x^{-1},$$

and in addition a one-parameter family of solutions of type II(a) of the asymptotic form $y \sim cx^b$, where c is any non-zero constant and

$$b = \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + a}.$$

There are no solutions of a lower order of magnitude.

3. Restriction of solutions to types I and II. Before establishing properties of type I and type II solutions I give conditions under which the classification is exhaustive.

THEOREM 1. *Let $g(x)$ be expressible as the sum of $g_1(x)$ and $g_2(x)$, where $g_1(x)$ is continuous for $x \geqslant 0$ and $g_2(x)$ continuously differentiable for $x \geqslant 0$, and*

3.1 $$\int_0^\infty |g_1|\,dx < \infty, \quad \int_0^\infty |g_2'|\,dx < \infty, \quad g_2(\infty) = 0.$$

Then 1.3 *has solutions of type I and II at most. Those of type I will certainly exist, and will include all solutions with an initial lower bound of the form* $|y(0)| + |y'(0)| >$ const. > 0. *In particular, all solutions are bounded.*

It should be remarked that in the linear case the same conditions ensure the validity of certain asymptotic integration formulae (**1**; **9**; **11**).

I prove first that r is bounded, which will prove the last statement of the theorem, and will also preclude the eventuality of a solution becoming infinite for a finite x-value. If r were not bounded for some solution, we could for any sufficiently large $A > 0$ find $x_1, x_2 \geqslant 0$ with

3.2 $$r(x_1) = A, \quad r(x_2) = 2A, \quad A \leqslant r(x) \leqslant 2A \text{ for } x_1 \leqslant x \leqslant x_2.$$

Now from 1.3 we have

3.3 $$\begin{aligned}(r^{2n} + ng_2y^2)' &= -2nyy'g_1 + ng_2'y^2 \\ &= O(A^{n+1}|g_1| + A^2|g_2'|),\end{aligned}$$

in (x_1, x_2) and hence, taking A so large that $A^{2n-2} > 2n \max|g_2|$, we have

$$(d/dx)\log(r^{2n} + ng_2y^2) = O(A^{1-n}|g_1| + A^{2-2n}|g_2'|),$$

in (x_1, x_2). We integrate this over (x_1, x_2), getting

$$[\log(r^{2n} + ng_2y^2)]_{x_1}^{x_2} = O\left(A^{1-n}\int_{x_1}^{x_2}|g_1|\,dx + A^{2-2n}\int_{x_1}^{x_2}|g_2'|\,dx\right).$$

We now make A increase without limit, and therewith x_1, x_2 if necessary. Since $g_2 \to 0$, the left-hand side tends to $2n \log 2$. The right, however, tends to zero as $A \to \infty$, by 3.1. This contradiction shows the boundedness of r, and y, y' also.

To deduce that r tends to a finite limit, we remark that it now follows from 3.1 that the right of 3.3 is absolutely integrable over $(0, \infty)$. This shows that $(r^{2n} + ng_2y^2)$ tends to a limit as $x \to \infty$, and hence r also, since $g_2 \to 0$. The solutions are therefore of types I and II at most.

To complete the proof of the theorem, it will be sufficient to show that if $r(0)$ is sufficiently large, then $r(\infty) \neq 0$. Suppose then that $r(\infty) = 0$, and write $r(0) = B$; we show that this gives a contradiction if B is sufficiently large. We must be able to find x_3, x_4 such that

$$r(x_3) = B, \quad r(x_4) = \tfrac{1}{2}B, \quad \tfrac{1}{2}B \leqslant r(x) \leqslant B \text{ for } x_3 \leqslant x \leqslant x_4.$$

If further we take B so large that $(\frac{1}{2}B)^{2n-2} > 2n \max |g_2|$, we have, by the above reasoning,

$$[\log (r^{2n} + ng_2y^2)]_{x_3}^{x_4} = O\left(B^{1-n} \int_{x_3}^{x_4} |g_1|\, dx + B^{2-2n} \int_{x_3}^{x_4} |g_2'|\, dx\right).$$

If now we make B increase without limit, the left-hand side will tend to $-(2n \log 2)$, and the right-hand side to zero, which constitutes the required contradiction.

In the rest of this paper I consider cases of 1.3 in which $g(x)$ is either monotonic and tending to zero, or else absolutely integrable over $(0, \infty)$. In both of these cases Theorem I shows that the solutions are at most of types I and II, and possibly of type I only.

4. Restriction of solutions to type I. I now give some simple sufficient conditions for the solutions to be of type I only. I prove first

THEOREM 2. *Let $g(x)$ be positive and continuously differentiable for $x > 0$, and tend monotonically to zero as $x \to \infty$. Then all solutions of* 1.3 *are of type I.*

Supposing if possible that for a certain solution of 1.3 we have $r \to 0$, we use the result that, by 1.3,

4.1 $$(r^{2n} g^{-1} + ny^2)' = -g'r^{2n} g^{-1},$$

which shows here that the function $(r^{2n} g^{-1} + ny^2)$ is non-decreasing. Since this function is positive for $y \not\equiv 0$, it follows that there is a positive constant C such that $(r^{2n} g^{-1} + ny^2) > C$ for all $x > 0$. Since $y \to 0$ we have, for all sufficiently large x, $r^{2n} g^{-1} > \frac{1}{2}C$, so that

4.2 $$r^{-1} = O(g^{-1/(2n)}).$$

Also from 1.3, or from 4.1, we have

$$\begin{aligned}(d/dx) \log (r^{2n} + ngy^2) &= ng'y^2(r^{2n} + ngy^2)^{-1}\\ &= O(g'y^2r^{-2n})\\ &= O(g'r^{2-2n})\\ &= O(g'g^{-1+1/n})\end{aligned}$$

using the fact that $g > 0$ and also 2.1, 4.2. Here the right-hand side is absolutely integrable over $(0, \infty)$, since g tends monotonically to zero. This proves that the function $\log (r^{2n} + ngy^2)$ tends to a finite limit as $x \to \infty$, thus contradicting the hypothesis that $r \to 0$.

The result just proved suggests that type II solutions may be expected when g is negative and small at ∞. That g must not be too small is shown by

THEOREM 3. *Let $g(x)$ be continuous for $x > 0$ and such that*

4.3 $$\int_0^\infty x|g(x)|\, dx < \infty.$$

Then 1.3 *has type I solutions only.*

We re-write 1.3 in the form

4.4 $$y'' + g_1(x)y = 0,$$

where $g_1 = g + y^{2n-2}$. We show first that if y is a type II solution of 1.3, and 4.3 holds, then

4.5 $$\int_0^\infty x|g_1(x)|\,dx < \infty.$$

In view of 4.3 it will be sufficient to prove that

4.6 $$\int_0^\infty xy^{2n-2}\,dx < \infty,$$

or again

4.7 $$\int_0^\infty xr^{2n-2}\,dx < \infty.$$

Now by 1.3 we have $(r^{2n})' = -2ngyy'$, and so, using 2.1,

4.8 $$r'r^{n-2} = O(g).$$

Integrating 4.8 over (x, ∞) we have, assuming that $r(\infty) = 0$,

4.9 $$r^{n-1} = O\left(\int_x^\infty |g|\,dx\right),$$

a result which will later be improved to 7.1. We have then

$$xr^{2n-2} = O\left\{x\left(\int_x^\infty |g|\,dx\right)^2\right\} = O\left(\int_x^\infty |g|\,dx\right),$$

using 4.3, so that 4.7 will be true if we have

$$\int_0^\infty dx \int_x^\infty |g(t)|\,dt < \infty.$$

This however is easily seen to follow from 4.3. We have therefore proved that a type II solution of 1.3 is also a solution of 4.4, where g_1 satisfies 4.5.

However 4.4 has, subject to 4.5, no solutions which are $o(1)$ as $x \to \infty$, having in fact, as is well known, two fundamental solutions of the asymptotic forms $y_1 \to 1$, $y_2 \sim x$, as $x \to \infty$. The existence of such a y_1 may be shown, for example, by transforming 4.4 to an integral equation and using successive approximation; we may then take $y_2 = y_1 \int^x y_1^{-2}\,dx$. This completes the proof of Theorem 3.

That the criterion 4.3 is fairly precise is shown by the example

$$y'' + y^{2n-1} - ayx^{-b} = 0,$$

for which it shows that there are no type II solutions for $b > 2$; in the case $b = 2$ they exist, as noted in §2, for $n = 2$ and $a > 2$.

5. The non-oscillation of type II solutions. Having considered the existence and magnitude (see 4.9) of type II solutions, I now give a simple sufficient criterion for their non-oscillatory character.

THEOREM 4. *Let $g(x)$ be negative and continuously differentiable for $x > 0$, and let it tend monotonically to zero as $x \to \infty$. Then type II solutions of* 1.3, *if any, have no zeros.*

The result $(r^{2n} + ngy^2)' = ng'y^2$ here shows that the function $(r^{2n} + ngy^2)$ is non-decreasing. At a zero of y this function would be positive, assuming $y \not\equiv 0$, and so could only tend to a positive limit. On the other hand for a type II solution we should have to have $(r^{2n} + ngy^2) \to 0$. This proves the theorem.

We can also deduce that for a type II solution we must have $r^{2n} + ngy^2 < 0$, showing that type II solutions satisfy in this case the bound $r^{2n-2} < n|g|$. While this result gives the correct order of magnitude of type II solutions in some cases, for 2.2 for example, a later result gives a precise numerical coefficient.

6. The phase-variable. In order to obtain sharper results, including asymptotic formulae for type I solutions, I introduce the phase-variable. I define by $\psi(\theta)$ the solution of

6.1 $$d^2\psi/d\theta^2 + n\psi^{2n-1} = 0,$$

with the initial conditions

$$\psi(0) = 1, \quad d\psi(\theta)/d\theta|_{\theta=0} = 0.$$

In the case $n = 1$ this reduces to the cosine function, for $n = 2$ to the lemniscate function. In the general case it is a periodic function of period $4K$, where

6.2 $$K = \int_0^1 (1 - \psi^{2n})^{-\frac{1}{2}}\, d\psi.$$

Using the abbreviation ψ_θ for $d\psi(\theta)/d\theta$ we have also

6.3 $$\psi_\theta^2 + \psi^{2n} = 1.$$

I now form the first-order differential equation satisfied by θ, the phase-variable defined for a solution y of 1.3 by

6.4 $$y = r\,\psi(\theta), \quad y' = r^n n^{-\frac{1}{2}} \psi_\theta(\theta), \qquad r > 0,$$

in agreement with 2.1. This definition leaves θ uncertain to the extent of an arbitrary multiple of $4K$, which need only be chosen so that θ is a continuous function of x.

We have first

6.5 $$\begin{aligned}(y'y^{-n})' &= n^{-\frac{1}{2}}(\psi_\theta\,\psi^{-n})' \\ &= n^{-\frac{1}{2}}(\psi_{\theta\theta}\,\psi^{-n} - n\psi_\theta^2\,\psi^{-n-1})\,\theta' \\ &= -n^{+\frac{1}{2}}(\psi^{n-1} + \psi_\theta^2\,\psi^{-n-1})\,\theta' \\ &= -n^{+\frac{1}{2}}\,\psi^{-n-1}\,\theta',\end{aligned}$$

using 6.1 and 6.3. Also

$$
\begin{aligned}
(y'y^{-n})' &= y''y^{-n} - ny'^2\, y^{-n-1} \\
&= -y^{n-1} - gy^{1-n} - ny'^2\, y^{-n-1} \\
&= -r^{n-1}\,\psi^{n-1} - gr^{1-n}\,\psi^{1-n} - r^{n-1}\,\psi_\theta{}^2\,\psi^{-n-1} \\
&= -r^{n-1}\,\psi^{-n-1} - gr^{1-n}\,\psi^{1-n},
\end{aligned} \tag{6.6}
$$

using 1.3 and 6.3. Combining 6.5, 6.6 we have

$$\theta' = r^{n-1}\, n^{-\frac{1}{2}} + gr^{1-n}\,\psi^2, \tag{6.7}$$

the required differential equation.

As regards the amplitude-variable r we have

$$
\begin{aligned}
(r^{2n})' &= (y^{2n} + ny'^2)' = -2ngyy' \\
&= -2n^{\frac{1}{2}}\, gr^{n+1}\,\psi\psi_\theta,
\end{aligned}
$$

and so

$$r' = -n^{-\frac{1}{2}}\, r^{2-n}\, g\psi\psi_\theta. \tag{6.8}$$

7. A bound for type II solutions. As a final result for type II solutions I give the precise form of the bound 4.9 for their magnitude.

THEOREM 5. *Let $g(x)$ be continuous and absolutely integrable over* $(0, \infty)$. *Then type II solutions of* 1.3, *if they exist, satisfy the bound*

$$r^{n-1} \leq (n-1)(n+1)^{-(n+1)/(2n)} \int_x^\infty |g|\, dx. \tag{7.1}$$

That the constant factor on the right of 7.1 cannot in general be reduced is shown by the example

$$y'' + y^3 - 3yx^{-2} = 0, \quad n = 2, \quad y = x^{-1}, \quad r = 3^{\frac{1}{4}}x^{-1},$$

for which the equality sign in 7.1 holds.

From 6.3 it may be deduced that

$$|\psi\psi_\theta| \leq n^{\frac{1}{2}}(n+1)^{-(n+1)/(2n)}$$

and so, by 6.8,

$$(n-1)r^{n-2}|r'| \leq (n-1)(n+1)^{-(n+1)/(2n)}|g|.$$

Integrating over (x, ∞) and putting $r(\infty) = 0$, for a type II solution, we have

$$r^{n-1} \leq \int_x^\infty (n-1)\, r^{n-2}|r'|\, dx \leq (n-1)(n+1)^{-(n+1)/(2n)} \int_x^\infty |g|\, dx,$$

the result stated.

8. The oscillations of type I solutions. I now pass to the investigation of type I solutions of 1.3, which are comparable to the non-trivial solutions of 1.1, at least in the respect that they have, by definition, an asymptotically constant amplitude. It remains to compare the two sets of solutions in respect

of oscillatory properties. In this section I find a rough estimate for the density of the zeros of a type I solution of 1.3, in analogy to known results for the linear case.

THEOREM 6. *Let $g(x)$ be continuous and tend to zero as $x \to \infty$. Then a type I solution of* 1.3, *if such exist, must be oscillatory as $x \to \infty$; if $N(x)$ denotes the number of its zeros in* $(0, x)$, *then*

8.1 $$N(x) \sim A^{n-1} x n^{-\frac{1}{2}} (2K)^{-1},$$

where K is given by 6.2, *and $A = r(\infty)$ for the solution in question.*

Before proceeding to the proof I remark that the condition $g(x) \to 0$ by itself may well be insufficient to ensure the existence of type I solutions; this certainly applies in the linear case, where the equation $y'' + y + g(x)y = 0$ with $g(x) \to 0$ can have solutions of asymptotically large and small amplitudes, without any of asymptotically finite positive amplitude. Examples of this phenomenon are given by

$$y'' + y(1 + x^{-a} \cos 2x) = 0 \qquad (0 < a < 1),$$

and by

$$y'' + y(1 + x^{-a} \cos x) = 0 \qquad (0 < a < \tfrac{1}{2}).$$

To prove 8.1 we integrate 6.7 over $(0, x)$, getting

8.2 $$\theta(x) - \theta(0) = \int_0^x r^{n-1} n^{-\frac{1}{2}}\, dx + \int_0^x g r^{1-n} \psi^2\, dx$$

8.3 $$= I_1 + I_2.$$

Since $r \to A > 0$ we have

8.4 $$I_1 \sim A^{n-1} x n^{-\frac{1}{2}}, \quad I_2 = O\left(\int_0^x |g|\, dx\right) = o(x).$$

Furthermore, since zeros of ψ, and so of y, occur when θ is an odd multiple of K, we have

8.5 $$N(x) = \theta(x)/(2K) + O(1),$$

using also the fact that $\theta(x)$ is an increasing function of x in the neighbourhood of a zero of ψ. The results 8.3–8.5 yield the proof of 8.1 and so prove the theorem.

For similar arguments in the linear case and references to other work on the linear case I refer to my paper **(2)**.

9. Asymptotic solutions. Finally I prove an approximation formula for type I solutions of 1.3 in terms of solutions of 1.1. To simplify the argument I have imposed more severe restrictions than are actually necessary for the result. Some essential improvement in the argument would be required however to make the result of similar generality to that of Ascoli (**1**; see also **9** and **11**) for the linear case.

THEOREM 7. *Let $g(x)$ be continuously twice differentiable for $x > 0$ and tend with $g'(x)$ monotonically to zero as $x \to \infty$. Let also $\int_0^\infty g^2\,dx < \infty$. Then to each type I solution of* 1.3 *there correspond two constants A, B, with $A > 0$, such that as $x \to \infty$,*

9.1 $$y = A\psi\left(A^{n-1}\,x\,n^{-\frac{1}{2}} + c_n A^{1-n}\int_0^x g\,dx + B\right) + o(1),$$

where c_n is a constant dependent only on n, and $\psi(\theta)$ is as defined in §6. *A corresponding formula for y' may be obtained by formal differentiation.*

The result shows that the influence of the linear perturbation term becomes vanishingly small for solutions of large amplitude.

In 9.1 the constant A denotes $r(\infty)$, and in view of 6.4 it is only necessary to prove that

9.2 $$\theta(x) = A^{n-1}\,x\,n^{-\frac{1}{2}} + c_n A^{1-n}\int_0^x g\,dx + B + o(1),$$

as $x \to \infty$. By 8.2–3 it will be sufficient to prove that

9.3 $$I_1 = A^{n-1}\,x\,n^{-\frac{1}{2}} + B_1 + o(1),$$

9.4 $$I_2 = c_n A^{1-n}\int_0^x g\,dx + B_2 + o(1),$$

where B_1, B_2 are constants for the solution in question.

In order to approximate to θ it is first necessary to approximate to r; this difficulty does not arise in the linear case ($n = 1$), since the right of 8.2 is then independent of r. For a first approximation we use the result $(r^{2n}+ngy^2)' = ng'y^2$. Integrating over (x, ∞), we have

$$A^{2n} - r^{2n} - ngy^2 = O\left(\int_x^\infty |g'|\,dx\right) = O(g),$$

since g is monotonic. Since $|y| < r$ we deduce that

9.5 $$r = A + O(g).$$

Using now 9.5 to obtain the second approximation we have

9.6 $$(r^{2n} + ngy^2)' = ng'r^2\psi^2 = ng'A^2\psi^2 + O(gg').$$

Integrating 9.6 over (x, ∞) we get

9.7 $$A^{2n} - r^{2n} - ngy^2 = nA^2\int_x^\infty g'\psi^2\,dx + O(g^2),$$

using the fact that g is monotonic.

I now write $\psi^2(\theta) = c_n + \phi(\theta)$, where

$$c_n = (4K)^{-1}\int_0^{4K}\psi^2(\theta)\,d\theta,$$

and

$$\Phi(\theta) = \int_0^\theta \phi(\theta)\, d\theta,$$

so that $\Phi(\theta)$ is a periodic and so bounded function of θ. We have then

$$ngy^2 = ngA^2\psi^2 + O(g^2) = ngA^2c_n + ngA^2\phi + O(g^2),$$

and also

$$\int_x^\infty g'\psi^2\, dx = -c_n g + \int_x^\infty g'\phi\, dx.$$

Using these in 9.7 we obtain

9.8 $$A^{2n} - r^{2n} - ngA^2\phi = nA^2 \int_x^\infty g'\phi\, dx + O(g^2).$$

To estimate the integral on the right of 9.8 we use the fact that

$$1 = \theta' A^{1-n} n^{-\frac{1}{2}} + O(g),$$

which follows from 6.7 and 9.5. From this we deduce that

$$\begin{aligned}\int_x^\infty g'\phi\, dx &= \int_x^\infty g'\phi\, \theta' dx . A^{1-n} n^{-\frac{1}{2}} + O\left(\int_x^\infty |gg'|\, dx\right)\\ &= [g'\Phi]_x^\infty A^{1-n} n^{-\frac{1}{2}} - \int_x^\infty g''\Phi\, dx . A^{1-n} n^{-\frac{1}{2}} + O(g^2)\\ &= O(g') + O(g^2).\end{aligned}$$

From 9.8 we therefore have

$$A^{2n} - r^{2n} - ngA^2\phi = O(g') + O(g^2),$$

and so, finally, we have the required second approximation

9.9 $$r = A - \tfrac{1}{2}g\phi\, A^{3-2n} + O(g') + O(g^2).$$

We pass to proving 9.3, 9.4 and so completing the proof of Theorem 7. As regards 9.3 we have

$$\begin{aligned}I_1 &= \int_0^x r^{n-1} n^{-\frac{1}{2}}\, dx\\ &= \int_0^x (A^{n-1} - \tfrac{1}{2}(n-1)\, g\phi A^{1-n})\, dx + \int_0^x O(g')\, dx + \int_0^x O(g^2)\, dx.\end{aligned}$$

Here the last two integrals are by hypothesis absolutely convergent taken over $(0, \infty)$. The term involving $g\phi$ is treated in the same way as the term involving $g'\phi$ in 9.8. We have

$$\begin{aligned}\int_0^x g\phi\, dx &= A^{1-n} n^{-\frac{1}{2}} \int_0^x g\phi\theta'\, dx + \int_0^x O(g^2)\, dx\\ &= A^{1-n} n^{-\frac{1}{2}} \left\{ [g\Phi]_0^x - \int_0^x g'\Phi\, dx \right\} + \int_0^x O(g^2)\, dx,\end{aligned}$$

and here both the integrals and the integrated term are asymptotic to constants as $x \to \infty$.

As regards 9.4 we have, using 9.5,

$$I_2 = \int_0^x gr^{1-n}\,\psi^2\,dx = A^{1-n}\int_0^x g\psi^2\,dx + \int_0^x O(g^2)\,dx$$
$$= A^{1-n}\,c_n\int_0^x g\,dx + A^{1-n}\int_0^x g\phi\,dx + \int_0^x O(g^2)\,dx.$$

Here the last integral on the right tends to a constant by hypothesis, while the second integral on the right has just been shown to do so. This justifies 9.4, and so completes the proof of Theorem 7.

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ON THE RIEMANN DERIVATIVES FOR INTEGRABLE FUNCTIONS

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1. Introduction. The central difference of order s of the function $f(x)$, $\Delta_{2h}^{s} f(x)$, corresponding to a number $h > 0$, is defined inductively by the relations

$$\Delta_{2h}^{1} f(x) = f(x+h) - f(x-h), \quad \Delta_{2h}^{s+1} f(x) = \Delta_{2h}^{1}[\Delta_{2h}^{s} f(x)].$$

If the limit of the difference quotient

$$\lim_{h \to 0} (2h)^{-s} \Delta_{2h}^{s} f(x)$$

exists at the point x, it is called the sth Riemann derivative or the generalized sth derivative of $f(x)$ at the point x.

This paper deals with the following problem: What are the necessary and sufficient conditions in order that a given integrable function $f(x)$ be p.p. (almost everywhere) equal to an indefinite repeated integral of another function $g(x)$? The main result (Theorem 2) gives this condition in terms of the weak convergence of the difference quotient of $f(x)$ to $g(x)$.

In particular, in §3 we prove by an elementary but apparently powerful method, a theorem which contains the well-known proposition of Brouwer (**3** or **1** or **6**, p. 70) which states:

A. *If $f(x)$ is continuous for $a < x < b$ and*

$$\Delta_{2h}^{s} f(x) = 0, \qquad a < x - sh < x + sh < b,$$

then $f(x)$ is a polynomial of degree at most $(s-1)$ in (a, b).

In §4 we come to our main result mentioned above which in §5 we use to establish a certain type of extension of the following theorem of de la Vallée-Poussin (**12**, p. 274):

B. *If $f(x)$ is continuous in $[a, b]$ and has at every point of this interval a finite second Riemann derivative $g(x)$, with $g(x) \in L(a, b)$, then*

$$f(x) = \int_a^x dt_1 \int_a^{t_1} g(t_2)\, dt_2 + c_0 + c_1 x, \qquad a < x < b,$$

where c_0 and c_1 are constants.

This last theorem is fundamental in the uniqueness theory of trigonometrical series.

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Finally in §6 we state results due to Anghelutza, Marchaud, Popoviciu, and Reid which follow from our main theorem. We also consider in this section an application to generalized convex functions.

2. Preliminary results. Consider the space $L(a, b)$, that is, the space of functions which are Lebesgue integrable over (a, b). The distance between two elements $f, h \in L(a, b)$ is defined as

$$||f - h|| = \int_a^b |f(x) - h(x)|\, dx.$$

If $||f_n - f|| \to 0$ as $n \to \infty$, then $f_n(x)$ is said to be *convergent in the mean* to $f(x)$. If

(i) $$||f_n|| < M, \text{ all } n$$

(ii) $$\int_a^x f_n(t)\, dt \to \int_a^x f(t)\, dt$$

for every $x \in [a, b]$, then $f_n(x)$ is said to be *weakly convergent* to $f(x)$ (with index 1).

It is known that convergence in the mean implies the weak convergence of $f_n(x)$ to $f(x)$ in the space $L(a, b)$.

We also define the space $L\{a, b\}$ as the class of functions integrable over every closed subinterval contained in the open interval (a, b). Let $f(x) \in L\{a,b\}$. Define the operator

$$A_h^1 f(x) = \frac{1}{2h}\int_{-h}^{h} f(x + t)\, dt = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\, dt, \qquad a < x - h < x + h < b, \tag{1}$$

and in general,

$$A_h^{s+1} f(x) = A_h^1\,[A_h^s f(x)].$$

These integral operators, or repeated average values, or integral means as they are sometimes called have been employed previously (**8** or **4**) and several of their properties necessary for our work will be recalled.

Lemma 1. *If $f(x) \in L\{a, b\}$, the operator $A_h^s f(x)$ is continuous and has derivatives $[A_h^s f(x)]^{(i)}$ $(i = 1, 2, \ldots, s - 1)$, which are absolutely continuous and moreover,*

$$[A_h^s f(x)]^{(s)} = (2h)^{-s}\Delta_{2h}^s f(x) \qquad p.p. \text{ in } (a - sh, b + sh). \tag{2}$$

Lemma 2. *If $f(x) \in L\{a, b\}$, then*

$$\lim_{h\to 0} A_h^s f(x) = f(x) \qquad (s = 1, 2, \ldots)$$

p.p. on (a, b).

Lemma 3. *If $f(x) \in L\{a, b\}$,*

$$\lim_{h\to 0} \int_\alpha^\beta |A_h^s f(x) - f(x)|\, dx = 0 \qquad (s = 1, 2, \ldots)$$

for every closed subinterval $[\alpha, \beta]$ contained in (a, b).

LEMMA 4.

$$A_{h_2}^{s_2} \Delta_{2h_1}^{s_1} f(x) = \Delta_{2h_1}^{s_1} A_{h_2}^{s_2} f(x),$$

$$a < x - s_1h_1 - s_2h_2 < x + s_1h_1 + s_2h_2 < b.$$

This relation follows readily from the linearity of the operator defined in (1) and since

$$\Delta_{2h}^{s} f(x) = \sum_{j=0}^{s} (-1)^j \binom{s}{j} f[x + (s-2j)\,h].$$

The following lemma will also be of use and for the proof, one may see **(10, p. 73)**.

LEMMA 5. *If the sth derivative $f^{(s)}(x)$ exists at the point x, then*

$$\lim_{h\to 0} (2h)^{-s} \Delta_{2h}^{s} f(x) = f^{(s)}(x).$$

For the sake of brevity we put

$$\mathfrak{J}_c^s g(x) = \int_c^x dt_1 \int_c^{t_1} dt_2 \ldots \int_c^{t_{s-1}} g(t_s)\, dt_s;$$

$P_s(x)$ always denotes a polynomial in x of degree not exceeding s.

3. An integral-difference equation. We shall now study an equation which connects the integral operators and the differences and in particular contains proposition A.

THEOREM 1. *Let $f(x)$ and $g(x)$ both $\in L\{a, b\}$. If, for every fixed h, $0 < h < (b-a)/2s$,*

$$\Delta_{2h}^{s} f(x) = (2h)^s A_h^s g(x) \tag{3}$$

for almost every x satisfying the inequality $a < x - sh < x + sh < b$, then there exists a P_{s-1} such that

$$f(x) = \mathfrak{J}_c^s g(x) + P_{s-1}(x) \tag{4}$$

almost everywhere in (a, b), with $a < c < b$.

Conversely, if the equality (4) *is satisfied almost everywhere, then the relation* (3) *holds almost everywhere.*

Proof. To begin with, consider the equation

$$\Delta_{2h}^{s} f(x) = 0 \tag{5}$$

holding for every fixed $0 < h < (b-a)/2s$ and for almost every $x \in (a + sh, b - sh)$.

To solve (5), consider first the case $f(x) \in C^{(s)}(a, b)$. Then, by Lemma 5

$$\lim_{h\to 0} (2h)^{-s} \Delta_{2h}^{s} f(x) = f^{(s)}(x), \qquad a < x < b,$$

which implies that $f^{(s)}(x) \equiv 0$ $(a < x < b)$, and consequently $f(x) = P_{s-1}(x)$ $(a < x < b)$.

Secondly, consider $f(x) \in L\{a, b\}$. Let k be fixed, such that $0 < k < (b - a)/(2s + 2)$. Applying the operator

$$A_k^{s+1}$$

to equation (5), by Lemma 4, we obtain

$$\Delta_{2h}^s A_k^{s+1} f(x) = 0$$

for every h and x such that

$$a + (s + 1)k < x - sh < x + sh < b - (s + 1)k.$$

Since

$$A_k^{s+1} f(x) \in C^{(s)},$$

we deduce from the first case that

$$A_k^{s+1} f(x) = P_{s-1}(x;\, k), \qquad x \in (a + (s + 1)k,\ b - (s + 1)k), \tag{6}$$

where the polynomial $P_{s-1}(x; k)$ depends on k.

Let $[\alpha, \beta]$ be a closed subinterval of (a, b). It is obvious that (6) implies that

$$A_{1/n}^{s+1} f(x) = P_{s-1}(x;\, 1/n)$$

for $\alpha < x < \beta$ and $n > N$, $N = N(\alpha, \beta)$. By Lemma 2,

$$A_{1/n}^{s+1} f(x)$$

approaches $f(x)$ p.p. in $[\alpha, \beta]$ as $n \to \infty$ and therefore $P_{s-1}(x; 1/n)$ must converge to a limit $P_{s-1}(x)$ p.p. in $[\alpha, \beta]$. This latter limit must be a polynomial of degree at most $s - 1$ for if a sequence of polynomials, the degree of each being at most l, converges for $l + 1$ different values of x, it converges for every value of x and its limit is a polynomial of degree at most l. Consequently,

$$f(x) = P_{s-1}(x) \qquad \text{p.p. in } [\alpha, \beta]. \tag{7}$$

Since the relation (7) holds for every closed subinterval of (a, b), it holds for (a, b).

Let us now return to equation (3) with the conditions specified in the theorem. It can easily be established (by induction on s) that

$$\Delta_{2h}^s \mathfrak{J}_c^s g(x) = (2h)^s A_h^s g(x), \tag{8}$$

and therefore the function

$$F(x) = f(x) - \mathfrak{J}_c^s g(x)$$

satisfies the equation (5) for $x \in (a + sh, b - sh)$. The theorem now follows readily. The converse is obvious.

In the particular case $f(x)$ is continuous in (a, b) and $g(x)$ is zero we obtain proposition A. The known proofs of the former proposition (for references see §1) that the authors have seen appear to depend rather too heavily on intrinsic properties of the differences and thus perhaps cannot be applied to the type of problems we consider in this paper.

4. The fundamental theorem.

Put

$$\sigma_n(x) = \tau_n(x) - g(x) \tag{9}$$

where

$$\tau_n(x) = (2h_n)^{-s}\Delta^s_{2h_n} f(x).$$

THEOREM 2. *Let $f(x)$ and $g(x) \in L\{a, b\}$. There exists a polynomial $P_{s-1}(x)$ such that*

$$f(x) = \mathfrak{J}^s_c\, g(x) + P_{s-1}(x)$$

p.p. in (a, b) with $a < c < b$, if and only if there exists a sequence $\{h_n\}$ of positive numbers converging to zero such that the sequence of functions $\{\tau_n(x)\}$ converges weakly to $g(x)$ in every closed subinterval $[\alpha, \beta]$ in (a, b), in other words if the conditions

$$\text{(i)} \quad \int_\alpha^\beta |\tau_n(x)|\, dx < M, \text{ all } n$$

$$\text{(ii)} \quad \int_\alpha^\beta \tau_n(x)\, dx \to \int_\alpha^\beta g(x)\, dx$$

are satisfied in every $[\alpha, \beta]$ of (a, b).

Proof. We shall at first prove the sufficiency of our hypothesis.

Let $[\alpha, \beta]$ be an arbitrary subinterval in (a, b) and let h be a fixed number with $0 < h < (\beta - \alpha)/2s$. Applying the operator A^s_h to both sides of the relation (9), for $x \in [\alpha + sh, \beta - sh]$ and sufficiently small h_n we find that

$$A^s_h\, \sigma_n(x) = (2h_n)^{-s}\Delta^s_{2h_n} A^s_h f(x) - A^s_h\, g(x) \tag{10}$$

where we have used Lemma 4 to invert the integral and difference operators of the first term on the right-hand side of this equality.

Now

$$A^1_h\, \sigma_n(x) = \frac{1}{2h}\int_{x-h}^{x+h} \sigma_n(t)\, dt = \frac{1}{2h}\int_{x-h}^{x+h} [\tau_n(t) - g(t)]\, dt$$

and hence by (ii), $A^1_h\, \sigma_n(x)$ converges to zero for $x \in [\alpha + h, \beta - h]$. But by (i),

$$|A^1_h\, \sigma_n(x)| < \frac{1}{2h}\int_\alpha^\beta |\sigma_n(t)|\, dt < \frac{1}{2h}\left[M + \int_\alpha^\beta |g(t)|\, dt\right],$$

and so as $g(x) \in L\{a, b\}$,

$$A^1_h\, \sigma_n(x)$$

converges dominatedly to zero for $x \in [\alpha + h, \beta - h]$. By Lebesgue's theorem on dominated convergence,

$$A^2_h\, \sigma_n(x) = \frac{1}{2h}\int_{x-h}^{x+h} A^1_h\, \sigma_n(t)\, dt$$

converges to zero for $x \in [\alpha + 2h, \beta - 2h]$.

Repeating this argument $s - 2$ more times we finally find that

$$\lim_{n\to\infty} A_h^s\, \sigma_n(x) = 0 \tag{11}$$

for $x \in [\alpha + sh, \beta - sh]$.

On the other hand, by Lemmas 1 and 5, and the relation (2), we deduce

$$\begin{aligned}\lim_{n\to\infty} (2h_n)^{-s}\Delta_{2h_n}^s A_h^s f(x) &= [A_h^s f(x)]^{(s)} \\ &= (2h)^{-s}\Delta_{2h}^s f(x)\end{aligned} \tag{12}$$

for almost every x in $[\alpha + sh, \beta - sh]$.

The relations (10), (11) and (12) show that

$$(2h)^{-s}\Delta_{2h}^s f(x) = A_h^s\, g(x)$$

for almost every $x \in [\alpha + sh, \beta - sh]$ and since $[\alpha, \beta]$ was an arbitrary closed subinterval of (a, b), the last condition holds for every fixed $0 < h < (b - a)/2s$ and for almost every x such that $a < x - sh < x + sh < b$. Applying Theorem 1, we deduce for almost every x in (a, b),

$$f(x) = \mathfrak{J}_c^s\, g(x) + P_{s-1}(x)$$

where c is fixed with $a < c < b$.

To establish the converse, we note that

$$\int_\alpha^\beta |\tau_n(x) - g(x)|\, dx \leq \int_\alpha^\beta |\tau_n(x) - A_{h_n}^s g(x)|\, dx + \int_\alpha^\beta |A_{h_n}^s g(x) - g(x)|\, dx,$$

where the first term on the right-hand side is zero by (8) and the second approaches zero by Lemma 3. Hence $\tau_n(x)$ converges in the mean and therefore weakly converges to $g(x)$ in $[\alpha, \beta]$. The theorem is now complete.

It is obvious from the proof that in case $s = 1$ the hypothesis (i) of the above theorem is not necessary.

THEOREM 3. *Let $f(x)$ and $g(x) \in L\{a, b\}$. The existence of a sequence of positive numbers $\{h_n\}$ converging to zero such that the sequence of functions $\{\tau_n(x)\}$ converges in the mean to $g(x)$ in every $[\alpha, \beta]$ of (a, b), is a necessary and sufficient condition in order that there exists a $P_{s-1}(x)$ with*

$$f(x) = \mathfrak{J}_c^s\, g(x) + P_{s-1}(x)$$

for almost every x in (a, b) where $a < c < b$.

Since mean convergence implies weak convergence, this theorem follows from the preceding.

5. Riemann derivatives. We now wish to express the previous theorem more directly in terms of the Riemann derivatives in a form, which, though weaker, can easily be recognized.

THEOREM 4. *Let* $f(x) \in L\{a, b\}$. *If*

(i) *there exists a sequence of positive numbers* $\{h_n\}$ *converging to zero such that*

$$\lim_{n\to\infty} \tau_n(x) = g(x) \qquad p.p. \text{ in } (a, b),$$

(ii) *there exists a function* $\tau(x) \in L\{a, b\}$ *such that*

$$\sup_{n>0} |\tau_n(x)| < \tau(x), \qquad a < x - sh_n < x + sh_n < b,$$

then there exists a $P_{s-1}(x)$ *such that*

$$f(x) = \mathfrak{I}_c^s g(x) + P_{s-1}(x)$$

p.p. in (a, b), *where* $a < c < b$.

Proof. By Lebesgue's theorem on dominated convergence, we obtain $g(x) \in L\{a, b\}$ and

$$\int_\alpha^\beta |\tau_n(x) - g(x)|\, dx \to 0, \qquad n \to \infty,$$

for every $[\alpha, \beta]$ in (a, b). The theorem now follows from the above.

In the particular case of Theorem 4, when $f(x)$ is continuous, conditions (i) and (ii) remaining unaltered, it follows immediately that for every x in (a, b),

$$f(x) = \mathfrak{I}_c^s g(x) + P_{s-1}(x).$$

Theorems 2, 3 or 4 may be considered as certain types of extensions of proposition B of §1 on the second Riemann derivatives to those of higher order, but we must note that the convergence conditions are somewhat different. The direct generalization (in case of an open interval) would be the following:

C. *If* $f(x)$ *is continuous in* (a, b) *and* $f^{(s-2)}(x)$ *exists everywhere in* (a, b), $f(x)$ *has a finite sth Riemann derivative* $g(x)$, *with* $g(x) \in L\{a, b\}$, *then for* $a < x < b$

$$f(x) = \mathfrak{I}_c^s g(x) + P_{s-1}(x).$$

For $s = 3, 4$, this result is known (**9** or **11**). It is conjectured that the result would hold for $s > 5$. That one must assume the existence of $f^{(s-2)}(x)$ for $s > 3$ even in the case $g(x) = 0$ can be seen from the following counter-example:

$$f(x) = |x| x^{s-3}.$$

In fact, the first $s - 3$ ordinary derivatives of this function exist, but the $(s - 2)$nd ordinary derivative does not exist at $x = 0$, while the Riemann sth derivative is everywhere zero.

The importance of proposition B lies in the fact that it is used in proving the result that if a trigonometrical series converges, except in an enumerable set,

to a finite and integrable function $g(x)$, then it is the Fourier series of $g(x)$ (**12**, p. 274).

6. Related theorems. We shall now state several corollaries to our theorems.

COROLLARY 1. *If $f(x) \in L\{a, b\}$ and $\tau_n(x)$ converges boundedly to $g(x)$ in every closed $[\alpha, \beta]$ of (a, b), then for almost every x in (a, b),*

$$f(x) = \mathfrak{J}_c^s g(x) + P_{s-1}(x).$$

COROLLARY 2. *If $f(x)$ is continuous in (a, b) and if*

$$\lim_{h \to 0} (2h)^{-s} \Delta_{2h}^s f(x) = g(x)$$

uniformly in every $[\alpha, \beta]$ of (a, b), then for every x in (a, b)

$$f(x) = \mathfrak{J}_c^s g(x) + P_{s-1}(x).$$

This corollary was previously established by Marchaud (**5**) and in the case $g(x) = 0$ by Anghelutza (**2**).

COROLLARY 3. *If $f(x) \in L\{a, b\}$ and*

$$\lim_{h \to 0} (2h)^{-s} \int_\alpha^\beta |\Delta_{2h}^s f(x)| \, dx = 0$$

for every $[\alpha, \beta]$ in (a, b), then

$$f(x) = P_{s-1}(x)$$

p.p. in (a, b).

This result is due to Reid (**8**), who used it to obtain integral criteria for a function to be p.p. equal to a solution of a linear differential equation.

Our final result concerns the class of functions defined in (a, b), every one of whose members can be represented in every $[\alpha, \beta]$ of (a, b) as the difference of two non-concave functions of order l. This class, which will be denoted by $DC^l\{a, b\}$, is connected with the class of functions of lth generalized bounded variation (**7**, p. 24).

At first we recall the definition of non-concave functions in general. A function $f(x)$ is said to be *non-concave of order l* in (a, b), if it is continuous in (a, b) and if for $a < x - lh < x + lh < b$,

(a) $$\Delta_{2h}^{l+1} f(x) \geqslant 0.$$

If $f(x)$ is non-concave of order l, it is known (**7**, pp. 48, 25) that

(b) $$\Delta_{2h}^l f(x) = O(h^l)$$

uniformly for x in every $[\alpha, \beta]$ of (a, b).

THEOREM 5. *Let $f(x) \in L\{a, b\}$ and $s > 2$. The necessary and sufficient conditions that there exists a $P_{s-1}(x)$ with*

$$f(x) = P_{s-1}(x) \qquad p.p. \text{ in } (a, b),$$

are

(i) *$f(x)$ is p.p. in (a, b) equal to a function $\phi(x) \in DC^{s-1}\{a, b\}$,*

(ii) $$\int_\alpha^\beta \Delta_{2h}^s f(x) = o(h^s), \qquad h \to 0,$$

for every $[\alpha, \beta]$ of (a, b).

Proof. The necessity is obvious.

To prove the sufficiency, according to Theorem 2 (the case $g(x) = 0$) we only need to show that the condition (i) implies that

$$\int_\alpha^\beta |\Delta_{2h}^s f(x)|\, dx = O(h^s)$$

for every $[\alpha, \beta]$ of (a, b).

It follows from the hypothesis (i) that $f(x) = \phi(x)$ p.p. in (a, b) where $\phi(x)$ can be represented in $[\alpha, \beta]$ in the form

$$\phi(x) = \phi_1(x) - \phi_2(x)$$

where $\phi_1(x)$ and $\phi_2(x)$ satisfy (a) and (b) for $l = s - 1$.

We have

$$\begin{aligned}
\int_\alpha^\beta |\Delta_{2h}^s f(x)|\, dx &= \int_\alpha^\beta |\Delta_{2h}^s\, \phi(x)|\, dx < \int_\alpha^\beta \Delta_{2h}^s\, \phi_1(x)\, dx + \int_\alpha^\beta \Delta_{2h}^s\, \phi_2(x) \\
&= \int_\alpha^\beta [\Delta_{2h}^{s-1}\, \phi_1(x + h) - \Delta_{2h}^{s-1}\, \phi_1(x - h)]\, dx \\
&\qquad + \int_\alpha^\beta [\Delta_{2h}^{s-1}\, \phi_2(x + h) - \Delta_{2h}^{s-1}\, \phi_2(x - h)]\, dx \\
&= \int_{\beta-h}^{\beta+h} \Delta_{2h}^{s-1}\, \phi_1(t)\, dt - \int_{\alpha-h}^{\alpha+h} \Delta_{2h}^{s-1}\, \phi_1(t)\, dt \\
&\qquad + \int_{\beta-h}^{\beta+h} \Delta_{2h}^{s-1}\, \phi_2(t)\, dt - \int_{\alpha-h}^{\alpha+h} \Delta_{2h}^{s-1}\, \phi_2(t)\, dt \\
&= O(h^s).
\end{aligned}$$

The theorem is now established.

The authors believe no previous attention has been given to results of this type, showing a relation between the class $DC^s\{a, b\}$ and polynomials of degree s.

Finally we would like to add that results corresponding to every one of the above theorems may be established for the forward and also the backward differences.

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LOGARITHMIC CAPACITY OF SETS AND DOUBLE TRIGONOMETRIC SERIES

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1. Introduction. It is the purpose of this paper to establish a closer connection between the logarithmic capacity of sets and double trigonometric series. In **(9)**, closed sets of logarithmic capacity zero were established as sets of uniqueness for a particular class of double trigonometric series under circular $(C, 1)$ summability. By slightly changing this class of series but still maintaining closed sets of logarithmic capacity zero as sets of uniqueness, it is shown in this paper that closed sets of positive logarithmic capacity form sets of multiplicity. Widening the class of series still further, it is shown here that closed sets of uniqueness and closed sets of logarithmic capacity zero also coincide for this new class under local uniform circular $(C, 1)$ summability.

The motivation for establishing these results arose from lectures on the uniqueness of one-dimensional trigonometric series delivered by Professor A. Beurling at the Institute for Advanced Study.

In this paper we are able, also, to obtain for planar sets a result analogous to one for linear sets given by Salem and Zygmund in **(8)**, where a necessary and sufficient condition that a linear set be of positive logarithmic capacity is given in terms of Fourier-Stieltjes series.

2. Definitions and Notation. Vectorial notation will be used whenever convenient and will be signified by capital letters thus:

$$P = (p, q), \quad X = (x, y), \quad \alpha X + \beta P = (\alpha x + \beta p, \alpha y + \beta q),$$
$$PX = px + qy, \quad |X| = (x^2 + y^2)^{\frac{1}{2}}.$$

Let E be a bounded Borel set. Then under the usual definition **(5**, p. 48**)**, E is said to be a set of positive logarithmic capacity if there exists a non-negative measure μ defined on the Borel sets in the plane such that $\mu(E) = 1$ and $\mu(A) = 0$ if $AE = 0$ and such that the potential

$$u(X) = \int_E \log|P - X|^{-1} d\mu(P) \tag{1}$$

has a positive upper bound. If no such measure exists for the set E, then E is said to be a set of logarithmic capacity zero.

It is known **(2)** that if E is a closed and bounded set of logarithmic capacity zero and D is a domain, then $D - DE$ is a domain. Furthermore if $g(X)$ is harmonic and bounded in $D - DE$, then there exists a function $h(X)$ harmonic in D and equal to $g(X)$ in $D - DE$ **(6**, p. 335**)**.

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A double trigonometric series

$$\sum_M a_M e^{iMX}, \tag{2}$$

where M represents a lattice point (m, n) and the a_M are arbitrary complex numbers will be said to converge circularly at the point X to the value $L(X)$ if the circular partial sums of rank R,

$$S_R(X) = \sum_{|M|<R} a_M e^{iMX}, \tag{3}$$

converge to the finite value $L(X)$. The series will be $(C, 1)$ circularly summable to $L(X)$ if the $(C, 1)$ circular means of rank R,

$$\sigma_R(X) = \sum_{|M|<R} a_M e^{iMX}\left(1 - \frac{|M|^2}{R^2}\right) = \frac{2}{R^2}\int_0^R S_r(X)\, r\,dr, \tag{4}$$

converge to the finite value $L(X)$.

In **(9)**, we called (2) a series of type (U) if $a_M = o(1)$, that is if $a_M \to 0$ as $|M| \to \infty$, and if the partial sums

$$\sum_{1<|M|<R} \frac{a_M}{|M|^2} e^{iMX}$$

converge uniformly. For the purpose of this paper it will be advantageous to widen the classes of series to be studied. We shall call (2) a series of class (U') if

$$\sum_{M\neq 0} \frac{a_M}{|M|^2} e^{iMX} \tag{5}$$

is the Fourier series of a continuous periodic function. We call (2) a series of class (B') if (5) is the Fourier series of a bounded function. For both of these classes no restriction is placed on the a_M.

The open disc of radius t and center P will be denoted in this paper by $D(P, t)$; the circumference of this disc, by $C(P, t)$. The fundamental semi-closed square

$$\{(x, y);\ -\pi < x \leqq \pi,\ -\pi < y \leqq \pi\}$$

will be designated by Ω; the interior of Ω by Ω°.

We say that the series (2) is locally uniformly $(C, 1)$ circularly summable in a set E if for every P in E, there exists a $D(P, t)$, $t > 0$, such that $\sigma_R(X)$ defined by (4) tends uniformly to a finite limit for X in $D(P, t)$.

Given a closed set $Z \subset \Omega$ we shall say that Z is a set of uniqueness for a series of class (U') under circular $(C, 1)$ summability if the fact that $\sum_M a_M e^{iMX}$ is a series of class (U') for which $\sigma_R(X) \to 0$ in $\Omega - Z$ implies that $a_M = 0$ for all M.

Given a closed set $Z \subset \Omega$, we shall say that Z is a set of uniqueness for series of class (B') under local uniform circular $(C, 1)$ summability if the fact that $\sum_M a_M e^{iMX}$ is a series of class (B') for which $\sigma_R(X) \to 0$ locally uniformly in $\Omega - Z$ implies that $a_M = 0$ for all M.

Let E be a bounded Borel set and let μ be a non-negative measure defined on the Borel sets of the plane. If $\mu(E) = 1$ and if $\mu(A) = 0$ for all Borel sets A with the property $AE = 0$, we say that μ is concentrated on E. Furthermore if E is contained in Ω, we can consider the Fourier-Stieltjes series of μ, written

$$d\mu \sim \sum a_M e^{iMX} \tag{6}$$

where

$$a_M = \frac{1}{4\pi^2} \int_\Omega e^{-iMX} \, d\mu(X).$$

3. Statement of main results. We shall prove the following three theorems connecting the logarithmic capacity of sets and double trigonometric series.

THEOREM 1. *Let E be a Borel set contained in the semi-closed square Ω. Then a necessary and sufficient condition that E be of positive logarithmic capacity is that there exists a non-negative measure μ concentrated on E whose Fourier-Stieltjes series is of class (B').*

THEOREM 2. *Let Z be a closed set contained in the semi-closed square Ω. Then a necessary and sufficient condition that Z be a set of uniqueness for series of class (U') under circular $(C, 1)$ summability is that Z be of logarithmic capacity zero.*

THEOREM 3. *Let Z be a closed set contained in the semi-closed square Ω. Then a necessary and sufficient condition that Z be a set of uniqueness for series of class (B') under local uniform circular $(C, 1)$ summability is that Z be of logarithmic capacity zero.*

Before proving these theorems, we should investigate the properties of Fourier-Stieltjes series and generalized Laplacians.

4. Fourier-Stieltjes series. Some of the notions in this section come from a course given by Professor Bochner at Princeton University.

Supposing $f(P)$ integrable on $C(X, t)$, we shall henceforth designate the mean-value of f on this circle by $f_X(t)$, thus

$$f_X(t) = \frac{1}{2\pi} \int_0^{2\pi} f(x + t\cos\theta, y + t\sin\theta) \, d\theta. \tag{7}$$

Then by (1), we have the following result:

LEMMA 1. *Let $f(x)$ be a function which is integrable on Ω and periodic of period 2π in each variable. Then the $(C, 1)$ circular mean of rank R of the Fourier series of $f(X)$ is given by*

$$\sigma_R(X) = 2 \int_0^\infty f_X(t) \, J_2(tR)/t \, dt \tag{8}$$

where $J_2(t)$ is the Bessel function of the first kind and order 2.

REMARK 1. (8) can be replaced by the equality

$$\sigma_R(X) = \frac{1}{\pi}\int_{E_2} f(X+P)\,\frac{J_2(|P|R)}{|P|^2}\,dP \tag{9}$$

where E_2 is the plane and the expression on the right side of (9) is understood to be the Lebesgue integral over E_2, where X in the integrand is a fixed point.

Remark 1 follows from the fact that for fixed X and R and for all $t > 0$ there is a constant K such that

$$\int_{D(0,\,t+1)-D(0,\,t)} |f(X+P)|\,dP < K(t+1),$$

$$\frac{|J_2(|P|R)|}{|P|^2} < K \qquad \text{for } |P| < 1,$$

$$\frac{|J_2(|P|R)|}{|P|^2} < \frac{K}{|P|^{5/2}} \qquad \text{for } |P| > 1.$$

For then

$$\int_{D(0,\,T)} |f(X+P)|\,\frac{|J_2(|P|R)|}{|P|^2}\,dP < \sum_{t=0}^{[T]} \int_{D(0,\,t+1)-D(0,\,t)} |f(X+P)|\,\frac{|J_2(|P|R)|}{|P|^2}\,dP$$
$$< K^2 + \sum_{t=1}^{[T]} \frac{K^2(t+1)}{t^{5/2}} < K_1,$$

where K_1 is another constant independent of T.

Given a non-negative measure μ concentrated on a Borel set E contained in the semi-closed square Ω we can form the $(C, 1)$ circular mean of rank R of its Fourier-Stieltjes series. It is clear, however, that we have to extend μ so that it is defined on the whole plane before we can get an expression similar to the right side of (9) for the $(C, 1)$ circular mean of rank R.

We handle the problem of the extension of μ defined in Ω in the following manner. Let η_M represent the point with the coordinates $(2\pi m, 2\pi n)$ where m and n represent any pair of integers positive, negative, or zero. Defining the point set $A + X$ to be the set of points $[P; P - X \text{ in } A]$, we have the double sequence of squares $\Omega_M = \Omega + \eta_M$. In particular, $\Omega_0 = \Omega$.

Now given a non-negative measure μ concentrated on a set $E \subset \Omega$, we call this measure μ_0 and define a measure μ_M for every M on the Borel sets of the plane by $\mu_M(A) = \mu(A - \eta_M)$. We thus see that μ_M is a non-negative measure concentrated on the set $E + \eta_M$. We then define a non-negative measure $\bar{\mu}$ on the bounded Borel sets of the plane by the formula

$$\bar{\mu}(A) = \sum_M \mu_M(A).$$

Noticing that

$$\bar{\mu}(A + \eta_{M_1}) = \sum_M \mu_M(A + \eta_{M_1}) = \sum_M \mu_{M-M_1}(A) = \bar{\mu}(A),$$

we call $\bar{\mu}$ the periodic extension of μ. Henceforth the Fourier-Stieltjes series of $\bar{\mu}$ will be understood to be the Fourier-Stieltjes series of μ as defined in §2.

With this extension of the measure, we are now in a position to state and prove the following lemma:

LEMMA 2. *Let μ be a non-negative measure concentrated on a Borel set E contained in Ω and let $\bar{\mu}$ be the periodic extension of μ. Let $\sigma_R(X)$ be the $(C, 1)$ circular mean of rank R of the Fourier-Stieltjes series of $\bar{\mu}$. Then*

$$\sigma_R(X) = \frac{1}{\pi}\int_{E_2} \frac{J_2(|P - X|R)}{|P - X|^2}\, d\bar{\mu}(P). \tag{10}$$

To prove the lemma, let us first observe that

$$\sigma_R(X) = \sum_{|M|<R} a_M\, e^{iMX}\left(1 - \frac{|M|^2}{R^2}\right) = \int_\Omega K_R(X - P)\, d\mu(P)$$

where

$$K_R(X) = \frac{1}{4\pi^2}\sum_{|M|<R} e^{iMX}\left(1 - \frac{|M|^2}{R^2}\right).$$

It is not difficult to see that the right side of (10) is a continuous function of X and that the same is true for $\sigma_R(X)$. Therefore to prove (10), it is only necessary to show that if A is any bounded Borel set then

$$\int_A dX \int_\Omega K_R(X - P) d\,\mu(P) = \pi^{-1}\int_A dX \int_{E_2} \frac{J_2(|P - X|R)}{|P - X|^2}\, d\bar{\mu}(P). \tag{11}$$

Now setting

$$\psi(B) = \pi^{-1}\int_B \frac{J_2(|X|R)}{|X|^2}\, dX$$

for any Borel set B, we see that ψ is an additive function of a set defined on the Borel sets in the plane. Furthermore, we see that the right side of (11) is by Fubini's theorem equal to $\int_{E_2}\psi(A - P)\, d\bar{\mu}(P)$ which in turn is equal to $\int_{E_2}\bar{\mu}(A - P)\, d\psi(P)$. This last fact follows from the observations that

$$\bar{\mu}(D(P, 1)) = 0(|P|^{\frac{1}{2}}), \quad \int_{E_2} |P|^{\frac{1}{2}}\, \frac{|J_2(|P|R)|}{|P|^2}\, dP < \infty$$

and an application of (**10**, Lemma 1). But $\bar{\mu}(A - P)$ is for fixed A, a bounded periodic function of P. Consequently by Remark 1,

$$\begin{aligned}
\int_{E_2}\bar{\mu}(A - P)\, d\psi(P) &= \int_\Omega \bar{\mu}(A - P)\, K_R(P)\, dP \\
&= \sum_M \int_\Omega \mu(A - P - \eta_M)\, K_R(P)\, dP \\
&= \sum_M \int_{\Omega_M} \mu(A - X)\, K_R(X)\, dX \\
&= \int_{E_2} \mu(A - X)\, K_R(X)\, dX.
\end{aligned}$$

However, letting $\chi_A(X)$ be the characteristic function of A, we have that

$$\int_{E_0} \mu(A - X)\, K_R(X)\, dX = \int_{E_0} d\mu(P) \int_{E_0} K_R(X)\, \chi_A(X + P)\, dX$$
$$= \int_\Omega d\mu(P) \int_{E_0} K_R(X - P)\, \chi_A(X)\, dX$$
$$= \int_\Omega d\mu(P) \int_A K_R(X - P)\, dX,$$

which is the left side of (11), and the lemma is proved.

LEMMA 3. *Given a non-negative measure μ concentrated on a Borel set E contained in Ω, let $\bar{\mu}$ be its periodic extension. Suppose $\bar{\mu}[D(X_0, t_0)] = 0$. Then the Fourier-Stieltjes series of $\bar{\mu}$ is uniformly circularly summable $(C, 1)$ to zero in $D(X_0, \frac{1}{2}t_0)$.*

For by (10), we have, since $\bar{\mu}[D(X_0, t_0)] = 0$, that

$$\sigma_R(X) = \frac{1}{\pi} \int_{E_2 - D(X_0, t_0)} \frac{J_2(|P - X|R)}{|P - X|^2}\, d\bar{\mu}(P).$$

However, there is a constant K such that

$$|J_2(u)| < K u^{-\frac{1}{2}} \qquad \text{for } u > 1.$$

Consequently, we see, for R sufficiently large and X in $D(X_0, \frac{1}{2}t_0)$, that

$$|\sigma_R(X)| < \frac{1}{\pi} \int_{E_2 - D(X_0, t_0)} \frac{K}{|P - X|^{5/2} R^{\frac{1}{2}}}\, d\bar{\mu}(P)$$
$$< \frac{1}{\pi R^{\frac{1}{2}}} \sum_{j=1}^{\infty} O(j^{-3/2}).$$

Therefore $\sigma_R(X) = O(R^{-\frac{1}{2}})$ uniformly for X in $D(X_0, \frac{1}{2}t_0)$, and the lemma is proved.

5. Generalized Laplacians. Let us suppose that $F(X)$ is defined and integrable in $D(X_0, t)$ and let us set

$$F_{1, X_0}(t) = \frac{1}{\pi t^2} \int_{D(X_0, t)} F(P)\, dP.$$

We then say that $F(X)$ has a generalized Laplacian of the second kind at the point X_0, designated by $\Delta_2 F(X_0)$, equal to α_1, if

$$\lim_{t \to 0} \frac{8[F_{1, X_0}(t) - F(X)]}{t^2} = \alpha_1.$$

For the purposes of this paper, it will be necessary to prove an extension of (9, Lemmas 1 and 2).

LEMMA 4. *Let $\sum a_M e^{iMX}$ be a double trigonometric series which is $(C, 1)$ circularly summable to zero at the point X_0. Furthermore, let*

$$- \sum_{M \neq 0} \frac{a_M}{|M|^2} e^{iMX}$$

be uniformly circularly summable $(C, 1)$ *to* $F(X) - \frac{1}{4}a_0|X|^2$ *in* $D(X_0, t_0)$, $t_0 > 0$. *Then* $\Delta_2 F(X_0) = 0$.

Setting

$$S_R(X) = \sum_{|M|<R} a_M e^{iMX} \text{ and } T_R(X) = -\sum_{1<|M|<R} \frac{a_M}{|M|^2} e^{iMX}\left(1 - \frac{|M|^2}{R^2}\right),$$

we observe that

$$\text{(a)} \quad \frac{1}{\pi t^2}\int_{D(X_0, t)} T_R(X)\, dX = -2\sum_{1<|M|<R} a_M e^{iMX_0} \frac{J_1(|M|t)}{|M|^3 t}\left(1 - \frac{|M|^2}{R^2}\right),$$

$$\text{(b)} \quad S_R(X_0) = o(R^2),$$

$$\text{(c)} \quad R^{-2}\sum_{1<|M|<R} a_M e^{iMX_0} \frac{J_1(|M|t)}{|M|t} = R^{-2}\int_0^R [S_u(X_0) - a_0]\frac{J_2(ut)}{u}\, du + R^{-2}[S_R(X_0) - a_0]\frac{J_1(Rt)}{Rt} = o(1) \text{ as } R \to \infty \text{ for fixed } t,$$

$$\text{(d)} \quad \frac{1}{\pi t^2}\int_{D(X_0, t)} T_R(X)\, dX \to F_{1,X_0}(t) - \frac{a_0}{4}[|X|_0^2 + \tfrac{1}{2}t^2] \text{ for } 0 < t < t_0.$$

We conclude from (a), (b), (c), and (d) that

$$\frac{8}{t^2}[F_{1,X_0}(t) - F(X_0)] = a_0 + \lim_{R\to\infty} 8\sum_{1<|M|<R} \frac{a_M e^{iMX_0}}{|M|^2 t^2}\left[1 - \frac{2J_1(|M|t)}{|M|t}\right].$$

The lemma then follows from (**3**, Theorem 1).

6. A particular set of Fourier coefficients. Let us set $\Phi(X) = 2\pi \log |X|^{-1}$ for X in Ω and then extend $\Phi(X)$ periodically; so that using the notation of §4, $\Phi(X) = 2\pi \log |X - \eta_M|^{-1}$ for X in Ω_M. We then have the following lemma:

LEMMA 5. *The Fourier coefficients* $1/\lambda_M$ of $\Phi(X)$, *with* $M \neq 0$, *have the following two properties*:

(i) $\dfrac{1}{\lambda_M} > 0$ *for all* M,

(ii) *There exists a constant* K *independent of* M *such that*

$$\left|\frac{1}{\lambda_M} - \frac{1}{|M|^2}\right| < K\left[\frac{1}{|M|^2(n^2+1)} + \frac{1}{|M|^2(m^2+1)}\right].$$

By means of Green's second identity, we observe that for $|M| \neq 0$,

$$(2\pi)^{-1}\int_{\Omega - D(0,\epsilon)} \log\frac{1}{|X|} e^{i(mx+ny)} dX = \frac{J_0(|M|\epsilon)}{|M|^2} - \int_{-\pi}^{\pi} \frac{\cos m\pi\, e^{iny} + \cos n\pi\, e^{imy}}{\pi^2 + y^2} dy + o(1).$$

Consequently, for $|M| \neq 0$,

$$\frac{1}{\lambda_M} = \frac{1}{|M|^2} - \frac{2}{|M|^2} \int_0^\pi \frac{\cos m\pi \cos ny + \cos n\pi \cos my}{\pi^2 + y^2} dy.$$

Since

$$\int_0^\pi \frac{dy}{\pi^2 + y^2} = \frac{1}{4},$$

we find $\lambda_M > 0$. As two integrations by parts show, there is a constant K such that

$$\left| \int_0^\pi \frac{\cos ny}{\pi^2 + y^2} dy \right| < \frac{K}{n^2 + 1}$$

for all n, and the lemma is proved.

7. Proof of Theorem 1. Let E be a Borel set contained in the semi-closed square Ω. Then a necessary and sufficient condition that E have positive logarithmic capacity is that there exists a non-negative measure μ concentrated on E such that

$$u(P) = \int_E \log |P - X|^{-1} d\mu(X) \tag{12}$$

is bounded above.

Using $\Phi(X)$ as defined in §6, we set

$$u_1(P) = \int_E \Phi(P - X)\, d\mu(X) \tag{13}$$

and observe that $u_1(P)$ is lower semi-continuous. Furthermore we observe that $u_1(P)$ is bounded above if and only if $u(P)$ is bounded above.

By (**4**, p. 84), if E has positive logarithmic capacity μ can be chosen so that $u(P)$ is continuous. But it is clear that $u(P)$ is continuous if and only if $u_1(P)$ is continuous.

To prove the sufficiency condition, let us suppose μ is concentrated on E and $d\mu \sim \sum a_M e^{iMX}$ and that

$$\sum_{M \neq 0} \frac{a_M}{|M|^2} e^{iMX}$$

is the Fourier series of a bounded function. Then it follows from Lemma 1 that

$$\left| \sum_{1<|M|<R} \frac{a_M}{|M|^2} e^{iMX} \left(1 - \frac{|M|^2}{R^2}\right) \right| < K,$$

where K is independent of R and X. But since $a_M = O(1)$, we conclude that the circular partial sums of rank R of

$$\sum_{M \neq 0} \frac{a_M}{|M|^2} e^{iMX}$$

are uniformly bounded. Furthermore the series

$$\sum_{M \neq 0} \frac{1}{|M|^2} \left[\frac{1}{n^2 + 1} + \frac{1}{m^2 + 1}\right]$$

is convergent. We thus have from Lemma 5 that the circular partial sums of rank R of

$$\sum_{M\neq 0} \frac{a_M}{\lambda_M} e^{iMX}$$

are uniformly bounded.

Let $u(P)$ and $u_1(P)$ be given by (12) and (13) respectively. Then for $|M| \neq 0$,

$$\frac{1}{4\pi^2}\int_\Omega e^{-iMP} u_1(P)\,dP = \frac{1}{4\pi^2}\int_E e^{-iMX} d\mu(X) \int_\Omega e^{-iM(P-X)}\Phi(P-X)\,dP = \frac{a_M}{\lambda_M} 4\pi^2.$$

But then $u_1(P)$ is an essentially bounded function which is lower semi-continuous and consequently bounded above. $u(P)$ is therefore bounded above and the sufficiency condition is proved.

To prove the necessity, let E be of positive logarithmic capacity. Let μ be a non-negative measure concentrated on E chosen so that $u(P)$ given by (12) is continuous. Consequently $u_1(P)$ given by (13) is a continuous periodic function. In the same manner as before, we find that the Fourier series of $u_1(P) - (4\pi^2)^{-1}\int_\Omega u_1(P)\,dP$ is

$$\sum_{M\neq 0} \frac{4\pi^2 a_M}{\lambda_M} e^{iMX},$$

where the a_M are the Fourier-Stieltjes coefficients of μ. But the $(C, 1)$ circular means of rank R of this series converge uniformly. It then follows from Lemma 5 that the $(C, 1)$ circular means of rank R of

$$\sum_{M\neq 0} \frac{a_M}{|M|^2} e^{iMX}$$

converge uniformly. This latter series is consequently the Fourier series of a continuous periodic function and the necessary condition is proved.

It is to be noticed that we have also proved the following fact which we state as a remark.

REMARK 2. Let E be a Borel set contained in the semi-closed square Ω. Then a necessary and sufficient condition that E be of positive logarithmic capacity is that there exists a non-negative measure μ concentrated on E whose Fourier-Stieltjes series is of class (U').

8. Proof of Theorem 2. Suppose that $T = \sum a_M e^{iMX}$ is $(C, 1)$ circularly summable to zero in $\Omega - Z$ where Z is a closed set of logarithmic capacity zero contained in the semi-closed square Ω and T is a series of class (U').

Set

$$F(X) - \tfrac{1}{4}a_0|X|^2 = -\lim_{R\to\infty} \sum_{1<|M|<R} \frac{a_M}{|M|^2}\left(1 - \frac{|M|^2}{R^2}\right) e^{iMX} \tag{14}$$

for all X in the plane. Since T is of class (U'), we have that the right side of (14) is uniformly convergent and consequently that

$$F(X) - \tfrac{1}{4}a_0|X|^2 = G(X) \tag{15}$$

where $G(X)$ is a continuous periodic function in the plane.

Take any bounded domain D in the plane. Then by Lemma 4 and the properties of sets of logarithmic capacity zero, we have that there exists a closed and bounded set of logarithmic capacity zero Z_1 such that $\Delta_2 F(X) = 0$ in the domain $D - DZ_1$. But by (**7**, p. 14), $F(X)$ is then harmonic in $D - DZ_1$. Since $F(X)$ is continuous in the closure of D, we obtain by (**6**, p. 335) a function $H(X)$ equal to $F(X)$ in $D - DZ_1$ and harmonic in D. But DZ_1 is of measure zero, $F(X)$ is therefore harmonic in D and consequently in the whole plane.

Furthermore, from (15), $F(X) = O(|X|^2)$. Therefore by (**11**, p. 19) $F(X)$ is a polynomial of at most degree 2, and the same is therefore true of $G(X)$. But $G(X)$ being continuous and periodic must then be a constant. Consequently

$$\sum_{1<|M|<R} \frac{a_M}{|M|^2} e^{iMX} \left(1 - \frac{|M|^2}{R^2}\right) \to K$$

uniformly for all X, where K is a constant. We conclude that $a_M = 0$ for $M \neq 0$, and then since our series was assumed $(C, 1)$ summable to zero in $\Omega - Z$, we have that $a_0 = 0$.

To show that Z is not a set of uniqueness if Z is a closed set of positive logarithmic capacity contained in the semi-closed square Ω, take a non-negative measure μ concentrated on Z with Fourier-Stieltjes series $\sum a_M e^{iMX}$ which is in class (U'). By Remark 2, this can always be done. By Lemma 3, $\sum a_M e^{iMX}$ is $(C, 1)$ circularly summable to zero in $\Omega - Z$. Z is therefore not a set of uniqueness, and the theorem is proved.

9. Proof of Theorem 3. Let us prove the sufficiency first. Suppose that $S_R(X)$ is given by (3) and $\sigma_R(X)$ by (4), and suppose, further, that $\sigma_R(X) \to 0$ locally uniformly in $\Omega - Z$ where Z is a closed set of logarithmic capacity zero contained in Ω. Let E_2 designate the plane and $\bar{Z} = \sum_M Z_M$ where $Z_M = Z + \eta_M$, η_M as in §4. Then $\sigma_R(X) \to 0$ locally uniformly in $E_2 - \bar{Z}$. It is furthermore clear that if $\sigma_R(X) \to 0$ uniformly in $D(X_0, t_0)$, then

$$-\sum_{1<|M|<R} \frac{a_M}{|M|^2} e^{iMX}$$

converges uniformly in $D(X_0, t_0)$.

Setting

$$F(X) - \tfrac{1}{4}a_0|X|^2 = -\lim_{R\to\infty} \sum_{1<|M|<R} \frac{a_M}{|M|^2} e^{iMX} \qquad \text{in } E_2 - Z, \tag{16}$$

we see that

(a) $\Delta_2 F(X) = 0$ in $E_2 - \bar{Z}$, by Lemma 4,

(b) $F(X)$ is continuous in $E_2 - \bar{Z}$ by the discussion in the above paragraph,

(c) $F(X) - \frac{1}{4}a_0|X|^2$ is bounded in $E_2 - \bar{Z}$ since $\sum a_M e^{iMX}$ is a series of class (B').

From the properties of $\bar{Z}$, (**7**, p. 14), and (a), (b), and (c), we conclude that there is a function $F_1(X)$ harmonic in E_2 and equal to $F(X)$ in $E_2 - \bar{Z}$.

Since $F_1(X) - \frac{1}{4}a_0|X|^2$ is bounded in $E_2 - Z$ and Z is of measure zero, $F_1(X) - \frac{1}{4}a_0|X|^2$ is bounded in E_2. But then $F_1(X)$ is $O(|X|^2)$ and consequently a polynomial of degree at most 2. Therefore $F_1(X) - \frac{1}{4}a_0|X|^2$ is a bounded polynomial; hence $F(X) - \frac{1}{4}a_0|X|^2$ is constant in $\Omega - Z$.

From (16) and the fact that our original series was in class (B'), we have that for $M \neq 0$

$$-\frac{a_M}{|M|^2} = \int_{\Omega - Z} [F(X) - \tfrac{1}{4}a_0X^2]\, e^{-iMX}\, dX.$$

We conclude first that $a_M = 0$ for $M \neq 0$ and then that $a_0 = 0$.

To prove the necessary condition of this theorem, let Z contained in Ω be a closed set of positive logarithmic capacity, and let μ be the non-negative measure of Theorem 1 which is concentrated on Z with Fourier-Stieltjes series $\sum a_M e^{iMX}$ which is in class (B'). By Lemma 3, this series is locally uniformly $(C, 1)$ circularly summable to zero in $\Omega - Z$. This completes the proof of the theorem.

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